Published by Institute of Physics Publishing for SISSA

Received: *April 7, 2005*
Accepted: *July 2, 2005*
Published: *July 22, 2005*

A paradigm of open/closed duality Liouville D-branes and the Kontsevich model

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Abstract: We argue that topological matrix models (matrix models of the Kontsevich type) are examples of exact open/closed duality. The duality works at finite N and for generic 't Hooft couplings. We consider in detail the paradigm of the Kontsevich matrix integral for two-dimensional topological gravity. We demonstrate that the Kontsevich model arises by topological localization of cubic open string field theory on N stable branes. Our analysis is based on standard worldsheet methods in the context of non-critical bosonic string theory. The stable branes have Neumann (FZZT) boundary conditions in the Liouville direction. Several generalizations are possible.

Keywords: String Field Theory, Topological Strings, D-branes.

Contents

1. Introduction and summary

The duality between open and closed strings is central to modern theoretical physics. It underlies, among other things, the relation between large-N gauge theories and closed strings [1]. Despite impressive progress, it is fair to say that we do not yet have a good conceptual grasp of this correspondence. Even by physics standards, we are quite far from a "proof" of AdS/CFT and related examples. We have little understanding of how general

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the gauge/gravity duality is, let alone how to generate the closed string dual of a given gauge theory. With this general motivation in mind, it is clearly of interest to develop exactly solvable models of open/closed duality. An important class of such models is offered by topological string theories, the paradigmatic example being the duality between Chern-Simons and the closed topological A-model [2].

Non-critical strings in low dimensions are another ideal context to sharpen our understanding of open/closed duality. Theories with $c \leq 1$ are fully solvable through the double-scaling limit of matrix models.[1] Indeed, the double-scaled matrix model for $c = 1$ strings has recently been re-interpreted [7] as the "open string field theory" for an infinite number of D0-branes. This provides another beautiful incarnation of exact open/closed duality. The doubled-scaled matrix model arises [8] as the worldvolume theory of the localized Liouville branes. These are the so-called "ZZ branes" [9], the unstable Liouville branes localized in the strong coupling region of the Liouville direction.[2]

Liouville theory admits also stable branes, the "FZZT" branes [14, 15], which are extended in the Liouville direction. What is the worldvolume theory on such extended branes?

Besides the well-known double-scaled matrix models, another, more mysterious, class of matrix models makes its appearance in low-dimensional string theories. The prototype of these models, which we shall collectively refer to as topological matrix models, is the Kontsevich cubic matrix integral [16], which computes the exact generating function of minimal $(2, 2k+1)$ matter coupled to gravity. Several other examples exist [17], covering a large class of $c \leq 1$ string theories.[3] These models deserve to be called topological because they compute certain topological invariants associated with the moduli space of Riemann surfaces [22, 16, 23−25, 21]. However, it must be noted that they actually contain all the information of the physical theories which are reached from the "topological point" by turning on deformations. As a result, any (p, q) bosonic string theory admits a polynomial matrix model *à la* Kontsevich which completely encodes its exact solution. Topological matrix models are treated in the usual 't Hooft expansion, with no double-scaling limit.

The reader will have guessed our punchline. Our basic contention is that topological matrix models generically arises in topological non-critical string theories as the open string field theory on N extended (FZZT) Liouville branes (tensored with an appropriate matter boundary state depending on the string theory under consideration). In this paper we work out in detail the prototype of the Kontsevich model. It is easy to envision that several generalizations should exist. We are going to argue that topological matrix models are examples of exact open/closed duality in very much the same spirit as the AdS/CFT correspondence. Perhaps the most interesting general lesson is that in this exactly solvable context we will able to precisely describe the mechanism by which a Riemann surface with boundaries is turned into a closed Riemann surface. Open string field theory [26]

[1] For reviews, see [3 − 6].

[2] A similar understanding is available for the double-scaled matrix models of $c < 1$ and $\hat{c} \leq 1$ theories [10 − 12]. See [13] for recent related work.

[3] The Penner model [18], the W_∞ model [19] and the normal matrix model [20] are particularly intriguing examples, related to $c = 1$ at the self-dual radius (see [21] for a recent review).

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on an infinite number of D-branes is seen to play a crucial role. Essentially the same mechanism is at work in the large-N transition for the topological A-model [27, 2, 28]. The Kontsevich integral offers an even more tractable case-study. Very recently, an interesting paper has appeared on the archive [29]. Building on previous work (e.g. [30]), these authors interpret topological matrix models as describing the dynamics of non-compact branes in the topological B-model for non-compact Calabi-Yau spaces. Although the language of [29] is very different from ours, there are clearly deep correspondences as well. Understanding in detail the relation between their point of view and ours should be an illuminating enterprise. Since the subject of topological matrix model may not be very widely known, and our explicit analysis will involve a few technicalities, in the rest of this introduction we review some background material and summarize our main conceptual points.

1.1 From open to closed worldsheets

It may be useful to begin by recalling the classic analysis [1] of the large-N limit of a gauge theory. In 't Hooft's double line notation, each gluon propagator becomes a strip, and gauge theory Feynman diagrams take the aspect of "fatgraphs", or open string Riemann surfaces, classified by the genus g and the number h of holes (boundaries). The generating functional for connected vacuum diagrams has then the familiar expansion (assuming all fields are in the adjoint),

$$\log \mathcal{Z}^{\rm close}(g_{\rm YM}, t) = \sum_{g=0}^{\infty} \sum_{h=2}^{\infty} (g_{\rm YM}^2)^{2g-2}\, t^h\, F_{g,h}\,, \qquad t \equiv g_{\rm YM}^2 N\,. \tag{1.1}$$

Nowadays we interpret this quite literally as the perturbative expansion of an open string theory, either because the full open string theory is just equal to the gauge theory (as e.g. for Chern-Simons theory [27]), or because we take an appropriate low-energy limit (as e.g. for $\mathcal{N} = 4$ SYM [31]). The general speculation [1] is that upon summing over the number of holes, (1.1) can be recast as the genus expansion for some *closed* string theory of coupling $g_s = g_{\rm YM}^2$. This speculation is sometimes justified by appealing to the intuition that diagrams with a larger and larger number of holes look more and more like smooth closed Riemann surfaces. This intuition is perfectly appropriate for the double-scaled matrix models, where the finite N theory is interpreted as a discretization of the closed Riemann surface; to recover the continuum limit, one must send $N \to \infty$ and tune t to the critical point t_c where diagrams with a diverging number of holes dominate.

However, in AdS/CFT, or in the Gopakumar-Vafa duality [2], t is a free parameter, corresponding on the closed string theory side to a geometric modulus. The intuition described above clearly goes wrong here. A much more fitting way in which the open/closed duality may come about in these cases is for *each* fatgraph of genus g and with h *holes* to be replaced by a closed Riemann surface of the same genus g and with h *punctures*: each hole is filled and replaced by a single closed string insertion. Very schematically, we may write

$$t \int d\rho\, \rho^{L_0}\, |\mathcal{B}\rangle_P \leftrightarrow t\mathcal{W}(P)\,. \tag{1.2}$$

Here the symbol $|B\rangle_P$ denotes the boundary state creating a hole of unit radius centered around the point P on the Riemann surface. Upon integration over the length of the boundary (indicated here by the modulus ρ), we can replace the boundary state with a closed string insertion $\mathcal{W}$ located at P. This idea is based on a correspondence between the moduli space of open surfaces and the moduli space of closed punctured surfaces which can be made very precise (see section 2 of [16]).

Clearly the position P in (1.2) is a modulus to be integrated over. Moreover, summing over the number of holes is equivalent to exponentiating the closed string insertion. As a result, we obtain the operator $\sim \exp(t \int d^2z\, \mathcal{W}(z))$, which implements a finite deformation of the closed string background. This is precisely what is required for the interpretation of t as a geometric parameter.

We were led to this viewpoint about open/closed duality, which probably has a long history (see e.g. [32, 33, 2, 28, 34 − 38]), by thinking about D-branes in imaginary time [36], where the mechanism (1.2) of boundaries shrinking to punctures can be described exactly.[4] In this paper we argue that topological matrix models are another very precise realization of this idea.

1.2 Review of $(2, 2k + 1)$ strings and the Kontsevich model

Minimal bosonic string theories are specified by a pair (p, q) of relatively prime integers.[5] In the continuum, they are formulated in the usual way by taking the total $\mathrm{CFT} = \mathrm{CFT}_{(p,q)} \oplus \mathrm{CFT}_{\mathrm{Liouville}} \oplus \mathrm{CFT}_{\mathrm{ghost}}$. Here $\mathrm{CFT}_{(p,q)}$ is a minimal (p, q) model [42], of central charge

$$c_{p,q} = 1 - 6\, \frac{(p-q)^2}{pq}\,. \tag{1.3}$$

The central charge of the Liouville CFT is of course chosen to be $26 - c_{p,q}$ to cancel the anomaly.

The $(2, 2k + 1)$ theories will be the focus of this paper. Perhaps the most familiar among these models is $(2, 3)$, which is pure two-dimensional quantum gravity ($c = 0$), or string theory embedded in one dimension. One way to find their complete solution is by the double-scaling limit of the one-matrix model, with the potential tuned to the multicritical point of order $k + 2$ [43]. Each of these theories has an infinite discrete set of physical closed string states, conventionally labeled as $\{\mathcal{O}_{2m+1}\}$, $m = 0, 1, 2, \cdots$. Observables are correlators of these operators, which is convenient to assemble in the following partition function, summed over all genera g,

$$\log \mathcal{Z}^{\mathrm{closed}}(g_s, t_n) = \sum_{g=0}^{\infty} g_s^{2g-2} \left\langle \exp\left(\sum_{n\,\mathrm{odd}} t_n \mathcal{O}_n \right) \right\rangle_g . \tag{1.4}$$

The partition functions for the different $(2, 2k + 1)$ theories are connected to each other by flows of the KdV hierarchy. This means that we simply need to expand $\mathcal{Z}^{\mathrm{closed}}(g_s, t_n)$

[4] A closely related viewpoint has been explained very clearly by Ooguri and Vafa [28], using a linear sigma-model perspective.

[5] See [3, 40, 39] for reviews and [41] for very recent progress in this subject.

around different background values of the sources t_n in order to obtain the correlators of the different $(2, 2k + 1)$ models. We choose our conventions so that $\{t_n = 0\,, \forall n\}$ corresponds to the $(2, 1)$ theory. Then correlators for $(2, 2k + 1)$ are found by perturbing around $t_n = \delta_{n,3} - \delta_{n,2k+3}$.

As first conjectured by Witten [22], the $(2, 1)$ model is equivalent to two-dimensional *topological* gravity [44 − 46], superficially a completely different theory. Topological gravity is a topological quantum field theory of cohomological type. In that context, the operators $\mathcal{O}_{2n+1}$ are interpreted as Morita-Mumford-Miller classes, certain closed forms of degree $2n$ on the moduli space of closed punctured Riemann surfaces; correlators $\langle \mathcal{O}_{k_1} \cdots \mathcal{O}_{k_n} \rangle_g$ are intersection numbers, topological invariants of this moduli space. An index theorem gives the selection rule

$$k_1 + \cdots + k_n = 6g - 6 + 3n \tag{1.5}$$

in order for the correlator to receive a non-zero contribution at genus g.

The remarkable equivalence of the $(2, 1)$ string theory with topological gravity was proved by Kontsevich [16], who found a combinatorial procedure to compute these intersection numbers. Kontsevich further recognized that his result for the partition function (1.4) could be efficiently summarized by the following matrix integral,[6]

$$\begin{aligned} \mathcal{Z}^{\text{closed}}(g_s, t) \;&=\; \rho(Z)^{-1} \int [dX] \, \exp\left(-\frac{1}{g_s} \, \mathrm{Tr} \left[\frac{1}{2} Z X^2 + \frac{1}{3} X^3 \right] \right), \\ \rho(Z) \;&\equiv\; \int [dX] \, \exp\left(-\frac{1}{2 g_s} \, \mathrm{Tr} \, Z X^2 \right). \end{aligned} \tag{1.6}$$

The integration is over the $N \times N$ *hermitian* matrix X. The matrix Z appearing in the quadratic term is another $N \times N$ hermitian matrix which encodes the dependence on the sources t_k through the dictionary

$$t_k = \frac{g_s}{k} \, \mathrm{Tr} \, Z^{-k} = \frac{g_s}{k} \sum_{n=1}^{N} \frac{1}{z_n^k} \qquad (k \text{ odd}), \tag{1.7}$$

where $\{z_n\}$ are the N eigenvalues of Z.

The Kontsevich integral works in a way which is truly miraculous — but which may also strike a familiar chord. The basic idea is an n-point closed string correlator

$$\langle \mathcal{O}_{k_1} \cdots \mathcal{O}_{k_n} \rangle_g \tag{1.8}$$

is extracted from the genus g vacuum amplitude *with n holes*. One can proceed perturbatively, using the obvious Feynman rules that follow from (1.6) (figure 1).

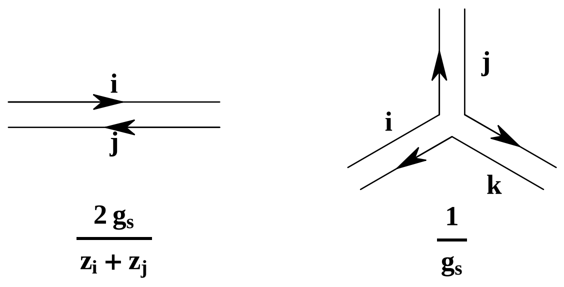

Figure 1: Feynman rules for the Kontsevich model.

[6]Of course, as written, the integral diverges. Analytic continuation $X \to iX$ makes the integral convergent for Z negative definite.

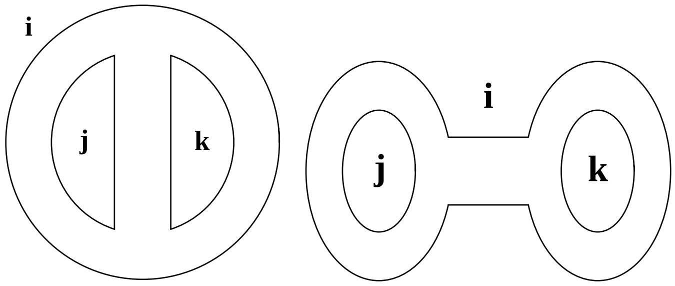

Figure 2: The two fatgraphs with $g = 0$ and $h = 3$. The indices i, j, k are Chan-Paton labels ranging from 1 to N. The sum of the two graphs is $g_s/(z_i z_j z_k)$. Upon summing over the Chan-Paton labels, this gives $t_1^3/(6g_s^2) \longrightarrow \langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\rangle_{g=0} = 1$.

Let us define $\Gamma_{g,n,N}$ to be the set of all connected fatgraphs of genus g, n holes, and a choice of a Chan-Paton index ranging from 1 to N for each hole (see examples in figure 2). The connected vacuum amplitude at genus g and with n holes is then

$$\mathcal{F}_{g,n,N} = g_s^{2g-2+n} \sum_{\gamma\in\Gamma_{g,n,N}} \frac{1}{\#\,\mathrm{Aut}(\gamma)} \prod_{(i,j)\in\gamma} \frac{2}{z_i + z_j}\,. \tag{1.9}$$

Individual Feynman diagrams give complicated rational expressions in the parameters $\{z_i\}$, but remarkably the total answer can always be expressed as

$$\mathcal{F}_{g,n,N} = g_s^{2g-2} \sum_{\{k_1\cdots k_n\}}^{k_i\ \mathrm{odd}} \frac{C_{\{k_1\cdots k_n\}}}{\#\,\mathrm{Aut}(k_1\cdots k_n)} \prod_{i=1}^{n} g_s \frac{\mathrm{Tr}\, Z^{-k_i}}{k_i}\,. \tag{1.10}$$

We see from the definition (1.7) that the parameters t_k play the role of generalized 't Hooft couplings. From (1.4), we recognize

$$\langle \mathcal{O}_{k_1}\cdots\mathcal{O}_{k_n}\rangle_g = C_{\{k_1\cdots k_n\}}\,. \tag{1.11}$$

The selection rule (1.5) is a simple consequence of Euler's theorem,

$$3(2-2g) = 3(\#V - \#P + n) = -\#P + 3n = -\sum_i k_i + 3n\,, \tag{1.12}$$

where $\#V$ and $\#P$ are the numbers of vertexes and propagators, and we used that $2(\#P) = 3(\#V)$. Notice that the selection rule (1.5) implies that at genus zero, all two-point correlators vanish. This gives a way to understand the prefactor $\rho(Z)^{-1}$ in the Kontsevich integral, which amounts to *removing* the fatgraphs with $g = 0$, $h = 2$ (the annuli) from the vacuum partition function of the matrix model.

In computing specific correlators using the Kontsevich integral, the rank N can be kept generic, as long as it is big enough to guarantee that the traces $\mathrm{Tr}\, Z^{-k}$ are functionally independent (otherwise the expression (1.10) is not uniquely defined); $N > \max(\{k_i\})/2$ suffices. If instead we are interested in the full partition function $\mathcal{Z}^{\mathrm{closed}}(g_s, t_k)$ for some fixed values of the infinitely many sources $\{t_k\}$, it is necessary to send $N \to \infty$ in order for the relation (1.7) to be invertible. So in particular we need infinite N to compute the correlators of the higher $(2, 2k+1)$ models, $k > 0$. Nevertheless, it makes perfect sense to keep N finite; the finite N Kontsevich model covers an N-dimensional submanifold in the moduli space of the closed string theory.

1.3 The Kontsevich model is cubic open string field theory

As we have just reviewed, the correlator of n closed string operators at genus g is computed in the Kontsevich model by the fatgraph vacuum amplitude of genus g and n boundaries. We propose that this is an exact open/closed duality: the Kontsevich model is to be interpreted as an open string field theory, dual to the $(2,1)$ bosonic closed string theory. The Kontsevich integral is to $(2,1)$ string theory as $\mathcal{N}=4$ SYM is[7] to IIB on $AdS_5 \times S^5$. The duality works just as explained in section 1.1. The closed string partition function $\mathcal{Z}^{\rm closed}(g_s, \{t_n\})$ is identified with the vacuum partition function $\mathcal{Z}^{\rm close}(g_s, \{z_i\})$ of the open string field theory. Each hole in the open description is replaced by the insertion of a closed string puncture, indeed, as we have emphasized in our review of the Kontsevich model, powers of the generalized 't Hooft couplings t_k count insertions of the closed string operator $\mathcal{O}_k$.

The reasoning that led Kontsevich to (1.6) uses the decomposition of the moduli space of Riemann surfaces [18, 47 − 50] that arises naturally in open string field theory [26] (OSFT), but so far this had not been given a direct physical interpretation. Here we are saying that in the Kontsevich model *is* OSFT. With the advantage of modern insight into the physics of D-branes, we can give a string theory "proof" of Kontsevich result. The logic is summarized by the following claims:

1. One can construct a family of stable D-branes in the $(2,1)$ string theory, labeled by a continuous parameter z.

2. Insertion of the boundary state $|\mathcal{B}(z)\rangle$ for any one such brane in a string amplitude is fully equivalent to the insertion of a closed string puncture, as in (1.2). In this case, the precise correspondence is

$$\int d\rho\, \rho^{L_0}\, |\mathcal{B}(z)\rangle_P \leftrightarrow \sum_{k\,\rm odd} \frac{\mathcal{O}_k(P)}{k\, z^k}\,. \tag{1.13}$$

3. The full cubic OSFT [26] on a collection of N of these D-branes, reduces precisely to the Kontsevich action (1.6). The parameters labeling the branes, $\{z_i\}$, $i = 1, \dots, N$, are the same as the parameters appearing in the quadratic term of the matrix integral.

These claims are sufficient to establish Kontsevich result. We just have to evaluate the string theory vacuum amplitude $\mathcal{Z}$ in the presence of N branes. We do this in two equivalent ways. Evaluating $\mathcal{Z}$ in the open channel, we have (claim 3) the sum of vacuum amplitudes

[7] An apparent difference is that in AdS/CFT the SYM theory is obtained only in the low-energy limit of the theory on the D3 branes in flat space, whereas the Kontsevich model is the full open string field theory. We take this as a small hint that a better way to understand AdS/CFT should exist, where the SYM theory *is* the full open string field theory of some appropriate branes. See section 7.

of the Kontsevich integral, $\mathcal{Z}^{\text{close}}(g_s, \{z_i\})$. Evaluating $\mathcal{Z}$ in the closed channel, we can replace each hole by a sum of closed string operators (claim 2), and obtain the generating function $\mathcal{Z}^{\text{closed}}(g_s, \{t_n\})$ of closed string correlators. This identifies the vacuum amplitude of the Kontsevich integral with the closed string partition function,

$$\mathcal{Z}^{\text{closed}}(g_s, \{t_n\}) \equiv \mathcal{Z}^{\text{close}}(g_s, \{z_i\}) \,, \tag{1.14}$$

which is what Kontsevich showed by more abstract and rigorous methods. The dictionary (1.2) between the "open parameters" $\{z_k\}$ and the "closed parameters" $\{t_k\}$ has its microscopic explanation in the rule (1.13) to replace a boundary with a specific closed string operator.[8]

1.4 Extended Liouville D-branes in topological string theory

Our goal is now to justify these claims by standard worldsheet methods. The $(2,1)$ string theory is strictly speaking outside the range of the definition given at the beginning of section 1.2, since the Kac table is empty and there is no $(2,1)$ "minimal" model. A possible definition is formal analytic continuation to $k \to 0$ of the double-scaling results [43], but this is unsatisfactory for our purposes. Fortunately, there are several other more intrinsic formulations, appearing to all yield the same results. Since $c_{2,1} = -2$, the simplest choice for the matter CFT is a pair of free, Grassmann odd scalars Θ^1 and Θ^2. This provides a continuum definition of the $(2,1)$ model as $c = -2$ matter coupled to $c = 28$ Liouville, and it is the set-up that we shall use in this paper. Sitting at the point $\{t_k = 0\}$ corresponds in particular to taking the bulk cosmological constant $\mu \equiv t_1 = 0$.[9]

Claim 1 is established by taking Dirichlet boundary conditions for the Θ^α and FZZT boundary conditions in the Liouville direction. The FZZT boundary state depends on a continuous parameter μ_B, the boundary cosmological constant, which can be thought of as the vev of the open string tachyon living on the brane. We identify $\mu_B = z$. The full boundary state is then

$$|\mathcal{B}(z)\rangle = |\mathcal{B}_\Theta^{\text{Dirichlet}}\rangle \otimes |\text{FZZT}(\mu_B = z) \otimes |\mathcal{B}_{\text{ghost}}\rangle \,. \tag{1.15}$$

FZZT boundary conditions are closely related to the notion of macroscopic loop operator $w(\ell)$ in two-dimensional quantum gravity [52, 53]. $w(\ell)$ is the operator that creates a hole of length ℓ in the Riemann surface, where the length is measured with the metric obtained by taking the Liouville field as the conformal factor. Then[10]

$$\int d\rho\, \rho^{L_0}\, |\mathcal{B}(z)\rangle \sim \int_0^\infty \frac{d\ell}{\ell}\, e^{-\ell z}\, w(\ell) \,. \tag{1.16}$$

[8] It makes sense to consider open string vacuum amplitudes at fixed values of $\{z_i\}$ because these are superselection parameters that do not fluctuate. This statement is dual to the statement that the closed string background $\{t_k\}$ is superselected [51].

[9] It may be useful to recall that in this theory (unlike the generic (p, q) model, $q \neq 1$) amplitudes depend analytically on μ and it makes sense to treat μ perturbatively.

[10] Here we are just tensoring the well-known relation between FZZT branes and macroscopic loops [52, 53, 14] with the (trivial) Dirichlet b.c. for the Θ^α.

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To obtain claim 2, we appeal to a standard bit of lore in non-critical string theory [52]. Under rather general conditions, the macroscopic loop operators can be expanded as $\ell \to 0$ as a sum of local closed string operators,

$$w(\ell) \sim \sum \ell^{x_k} \mathcal{O}_k \,, \tag{1.17}$$

where $x_k \geq 0$. A simple argument based on conservation of the Liouville momentum (section 3.1), fixes the exponents to be $x_k = 2k+1$. The $\ell \to 0$ expansion of $w(\ell)$ translates after Laplace transform (1.16) into a $z \to \infty$ expansion of $|\mathcal{B}(z)\rangle$ as a sum of terms $\sim z^{-2k-1}\mathcal{O}_k$. This gives claim 2, modulo fixing the precise normalization of the operators $\mathcal{O}_k$. In principle these normalization coefficients could be obtained by a very careful analysis of the boundary state, but it it easiest to determine them indirectly by consistency, as we explain in section 5. This replacement of a boundary with a sum of closed string insertions is a generic fact in low-dimensional string theory, and does not appear to depend on the topological nature of the $(2,1)$ model.

By contrast, claim 3 is based on a mechanism of topological localization, similar in spirit to the reduction of the open topological A-model on $T^*(S^3)$ to Chern-Simons theory on S^3 [27]. The worldsheet boundary CFT admits a nilpotent scalar supercharge Q_S [54], anti-commuting with the usual BRST operator Q_B. The open string (first-quantized) hamiltonian is a Q_S anti-commutator, so it can be rescaled by an overall constant without changing the physics. As in the case of [27], the only contributions to open string amplitudes come from the region of moduli space where the Riemann surfaces degenerate to ordinary Feynman graphs. In the usual OSFT decomposition of moduli space in terms of trivalent vertices and propagators (strips) of length $t^{(\alpha)}$, this is the limit in which each $t^{(\alpha)} \to \infty$. In this limit, the full cubic OSFT collapses to a cubic matrix integral for the open string "tachyon". A detailed analysis of Liouville BCFT correlators (section 4.3 and appendix) shows that this matrix integral is exactly the Kontsevich model, provided we identify the boundary cosmological constants $\{\mu_B^i\}$, $i = 1, \cdots, N$, with the parameters $\{z_i\}$.

The discussion has been phrased so far in terms of worldsheet ideas. An alternative powerful viewpoint is the use of "spacetime" Ward identities, which we briefly outline in section 5 of the paper. Finally the whole contruction admits an instructive generalization to non-zero bulk cosmological constant μ, as described in section 6.

2. Closed bosonic strings in $D = -2$

We define the $(2,1)$ closed string theory by choosing the total worldsheet action to be

$$S = S^{c=-2}_{\rm matter} + S^{c=28}_{\rm Liou} + S^{c=-26}_{\rm ghost} \,. \tag{2.1}$$

The matter CFT is that of a pair of real, *Grassmann odd* scalar fields $\Theta^1(z,\bar{z})$ and $\Theta^2(z,\bar{z})$, with the free action

$$S^{c=-2}_{\rm matter} = \frac{1}{2\pi}\int d^2z\, \epsilon_{\alpha\beta}\partial\Theta^\alpha\bar{\partial}\Theta^\beta \,, \qquad \alpha,\beta = 1,2 \,. \tag{2.2}$$

There is some freedom as to which CFT with $c = -2$ one should pick. Another possibility [54] would be to take the more familiar $\xi\eta$ ghost system, related to the Θ^α system as follows:

$$\eta(z) = \partial\Theta^2(z,\bar z)\,, \qquad \xi(z) + \xi(\bar z) = \Theta^1(z,\bar z)\,. \tag{2.3}$$

The two theories differ only in the treatment of the zero-modes. $\Theta^1(z,\bar z)$ has only one *non-chiral* zero-mode (the same is true for $\Theta^2(z,\bar z)$), so the mode expansion reads

$$\Theta^\alpha(z,\bar z) = \theta_0^\alpha + \frac{1}{2}\,d_0^\alpha \ln|z|^2 + \frac{1}{\sqrt{2}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{d_n^\alpha}{nz^n} + \frac{\bar d_n^\alpha}{n\bar z^n}\right). \tag{2.4}$$

This is a rather subtle difference, but we believe that the choice of the Θ^α is the correct one. First, this is the most obvious choice to describe "strings in minus two dimensions". It is indeed the choice singled out by defining the theory from double-scaling of a matrix model for random surfaces embedded in minus two dimensions [55 – 59]. Second, the treatment of closed string correlators is simpler, as unlike the $\xi\eta$ system, the Θ^α system does not require the introduction of screening charges. We come back to this point in the next subsection. Finally, this is the choice that will naturally lead to the Kontsevich model.

The Θ^α system has of course properties very similar to those of a pair of free bosons, one need only keep track of Grassmann minus signs. The OPE reads

$$\Theta^1(z,\bar z)\Theta^2(0) \sim -\frac{1}{2}\log|z|^2\,, \tag{2.5}$$

and the stress tensor is

$$T_\Theta = \epsilon_{\alpha\beta}\partial\Theta^\alpha\partial\Theta^\beta\,. \tag{2.6}$$

(Note that in this paper we set $\alpha' = 1$). The Θ^α CFT as an obvious global SL(2) invariance that rotates the fields. This symmetry does not extend to an affine symmetry but to a W_3 algebra [60].

It is amusing to check the modular invariance of the Θ^α system. The vacuum amplitude on the torus can be easily found by explicit computation of the trace,[11]

$$\mathrm{Tr}\left[(-1)^F\,\theta_0^1\theta_0^2\,q^{L_0+1/12}\,\bar q^{\bar L_0+1/12}\right] = 2\pi\tau_2\,|q|^{1/6}\prod_{n=1}^{\infty}|1-q^n|^4 = 2\pi\tau_2\,|\eta(\tau)|^4\,, \tag{2.7}$$

and is indeed modular invariant. The unusual factor of τ_2 is a consequence of the zero-mode insertions, while the $(-1)^F$ factor follows from odd-Grassmanality. As it should be, this is the inverse of the torus vacuum amplitude for two free bosons. We should also mention that (orbifolds of) Θ^α systems have been studied in detail [60] as prototypes of logarithmic CFTs [61, 62]. Liouville CFT has been well-understood in recent years (see e.g. [15, 14, 9] and references therein), and it is largely thanks to this progress that we shall be able to carry our analysis. We collect here some standard formulas:

$$S_{\mathrm{Liou}} = \frac{1}{2\pi}\int d^2z\left(\partial\phi\bar\partial\phi + Q\,R\,\phi + \mu\,e^{2b\phi}\right) \tag{2.8}$$

[11] To obtain a non-zero amplitude, we must of course insert the two zero modes θ_0^1 and θ_0^2.

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$$c_{\rm Liou} \equiv 1 + 6Q^2 \,, \qquad Q = b + \frac{1}{b} \tag{2.9}$$

$$\phi(z,\bar z)\phi(0) \sim -\frac{1}{2}\log|z|^2 \tag{2.10}$$

$$T^{\rm Liou} = -\partial\phi\partial\phi + Q\partial^2\phi \tag{2.11}$$

$$\mathcal{V}_\alpha \equiv e^{2\alpha\phi} \,, \qquad h_\alpha = \alpha(Q-\alpha)\,. \tag{2.12}$$

Specializing to $c_{\rm Liou} = 28$, we have $Q = 3/\sqrt{2}$, $b = 1/\sqrt{2}$. We shall keep the symbol b in many formulas to facilitate future generalizations; unless otherwise stated, it is understood that $b \equiv 1/\sqrt{2}$.

2.1 Remarks on closed string observables

In this subsection we offer some side remarks about closed string amplitudes. Our main interest is in the *open* string sector, indeed the essential point is that one can bypass the closed string theory altogether and compute everything using open string field theory (the Kontsevich model), which is structurally much simpler. The subject of closed string amplitudes in topological gravity is notoriously subtle [22, 46, 63 – 66]. Here we attempt to make contact with some of the previous work and suggest that the action (2.1), (2.2) may offer a different and simpler starting point.

A model very similar to (2.1), (2.2) (but with the $\xi\eta$ system instead of the Θ^α system) was considered by Distler, who observed that by an elegant change of variables (see (3.12) below) the bosonic $(2,1)$ theory could be formally related to the topological gravity formulation of [44]. This is one of the several [22, 46, 63 – 66] (closely related) field-theoretic formulations of topological gravity (see [67, 68] for reviews). They all have in common a sophisticated BRST machinery extending the ordinary moduli space to a (non-standard) super-moduli space, which in essence is just the space of differential forms over the bosonic moduli space. These formulations (as particularly transparent in Verlinde's set-up [46]) make it manifest that closed string amplitudes are intersection numbers on the moduli space. In this paper we will carry our analysis in the context of the bosonic $(2,1)$ theory, but we believe that an analogous derivation of the Kontsevich model must be possible in the BRST formulations of topological gravity.

A potential worry is the claim by Distler and Nelson [65] that the bosonic $(2,1)$ model (*with the $\xi\eta$ system*) does not correctly reproduce the topological gravity results, and that the full BRST machinery is necessary to obtain the correct measure of integration over the moduli space. It is quite difficult to compute topological gravity amplitudes from first principles using standard worldsheet methods, in any of the field-theoretic formulations. The difficulty stems from the very nature of the observables: amplitudes are naively zero before integration over the moduli space, and receive contributions only from "contact terms" (degenerations of the punctured surface). This is related to the fact that there are no non-trivial closed string states in the absolute BRST cohomology, the only observables being in the semi-relative cohomology.

However, the different zero-mode structure of the Θ^α system does certainly affect the calculation of these contact terms. We believe that a careful analysis using the action (2.2) would fully account for the correct contact term algebra. This is very plausible in light of the fact that using this worldsheet action we will obtain the Kontsevich model. More concretely, our derivation of the Kontsevich model suggests a "canonical" form for the closed string vertex operators,

$$\mathcal{O}_{2k+1} = e^{2(1-k)b\phi}\,\mathcal{P}_k(\partial\Theta^\alpha, \bar{\partial}\Theta^\beta)\, c\bar{c}\,. \tag{2.13}$$

Here $\mathcal{P}_k(\partial\Theta^\alpha, \bar{\partial}\Theta^\beta)$ is a primary of dimension $\left(\frac{k(k+1)}{2}, \frac{k(k+1)}{2}\right)$, and it should be invariant under the SL(2) symmetry. This follows from the fact that the D-branes which we use to obtain the Kontsevich model are SL(2) invariant. It turns out that there is a *unique* such operator in the Θ^α CFT. This can be seen from the results in [60]. In that paper it is proved that (in each chiral half of the theory), for each $j \in \mathbb{N}/2$, there is exactly one spin-j SL(2) multiplet of primaries, of conformal dimension $j(2j+1)$. Since there is only one way to combine the chiral and antichiral fields into an SL(2) singlet, this shows the uniqueness of $\mathcal{P}_k(\partial\Theta^\alpha, \bar{\partial}\Theta^\beta)$.

The operators (2.13) differ from the ones considered by Distler [54], which are not SL(2) invariant. In [54] a further operation of "picture changing" was necessary in order to obtain non-zero correlators. In that language, the operators (2.13) are already in the correct picture and in principle their correlators can be evaluated without any extra screening insertions. The only selection rule comes from anomalous conservation of Liouville momentum, and it is precisely (1.5).

3. Open string theory on stable branes

We now turn to the open string sector of the $(2,1)$ theory. The natural boundary conditions for the Θ^α system are either Neumann or Dirichlet. Boundary conditions for the Liouville CFT are either ZZ (unstable, localized at $\phi \to \infty$) or FZZT (stable, extended in the Liouville direction). The choice leading to the Kontsevich model is to combine Dirichlet b.c. for Θ^α and FZZT b.c. for Liouville,[12]

$$i(\partial\phi - \bar{\partial}\phi)|_\partial = 4\pi\mu_B\, e^{b\phi}\,, \qquad \Theta^\alpha|_\partial = 0\,. \tag{3.1}$$

The FZZT boundary conditions are generated by the adding to the Liouville action the boundary term

$$\mu_B \int_\partial e^{b\phi}\,. \tag{3.2}$$

One of the basic ingredients of our construction is the claim that amplitudes with boundaries can be reduced to amplitudes where each boundary is replaced by a specific closed string insertion. The same phenomenon was demonstrated for D-branes in imaginary time [36] through a precise CFT analysis in the usual framework of (critical) string theory. In the present case it is easiest to use instead the language of two-dimensional quantum gravity (or non-critical string theory). This language gives a very useful geometric understanding of the FZZT boundary state, which we now review.

[12] Another interesting choice is Neumann for Θ^α and ZZ for Liouville, related to the double-scaled matrix model, see section 7.1.

3.1 Macroscopic loops

In critical string theory, we are instructed to integrate the appropriate CFT amplitudes over the moduli space of Riemann surfaces. In quantum gravity, we integrate over the two-dimensional metric (modulo diffeomorphisms). Of course the two points of view are completely equivalent, as the integral over metrics can be replaced by the Liouville path-integral followed by integration over the moduli. Schematically,

$$\int \frac{[\mathcal{D}g_{ab}]}{\text{Diff}} \int [\mathcal{D}X]\,(\mathcal{O}_1 \cdots \mathcal{O}_n) \leftrightarrow \int_{\mathcal{M}_{g,n}} [d\,m] \int [\mathcal{D}X]\,[\mathcal{D}\phi]\,[\mathcal{D}b]\,[\mathcal{D}c]\,(\mathcal{O}_1 \cdots \mathcal{O}_n)\,. \tag{3.3}$$

Here $\mathcal{M}_{g,n}$ denotes the moduli space of closed Riemann surfaces of genus g and n punctures, ϕ the Liouville field, X a collective label for the matter fields, and $\{\mathcal{O}_k\}$ a generic assortment of local operators. To compute amplitudes in the presence of h boundaries, in the language of critical string theory we would of course integrate over the moduli space of $\mathcal{M}_{g,n,h}$ of Riemann surfaces with h holes, specifying appropriate boundary conditions for all the fields. In the language of quantum gravity, FZZT boundary conditions have the simple interpretation of introducing a "weight" for each boundary length ℓ_i [52, 14],

$$\int \frac{[\mathcal{D}g_{ab}]}{\text{Diff}}\, e^{-\sum_{i=1}^{h} \mu_B^i\, \ell_i[g]} \int [\mathcal{D}X]\,(\cdots) \equiv \left\langle \prod_i \left[\int \frac{d\ell_i}{\ell_i}\, e^{-\mu_B^i \ell_i} w(\ell_i)\right] \cdots \right\rangle . \tag{3.4}$$

Here on the r.h.s. we have introduced the definition of the macroscopic loop operator $w(\ell)$, which is the operator creating a boundary of length ℓ in the two-dimensional universe. Note that we have also left implicit a choice of boundary conditions for the matter fields X. Another standard object is the Laplace transform of $w(\ell)$,

$$W(\mu_B) \equiv \int \frac{d\ell}{\ell}\, e^{-\mu_B \ell} w(\ell)\,. \tag{3.5}$$

In the presence of three or more boundaries, each loop operator $w(\ell)$ can be expanded in non-negative powers of ℓ [52], or equivalently, $W(\mu_B)$ can be expanded in inverse powers of μ_B [52]. Each term in this expansion represents a local disturbance of the surface, and is thus equivalent to the insertion of a local operator.

In our case, the expansion will take the general form

$$W^{\text{Dirichlet}}(\mu_B) = g_s \sum_k^{\infty} c_k\, \frac{\mathcal{O}_k}{\mu_B^{x_k}}. \tag{3.6}$$

The superscript on W is a reminder that we are imposing Dirichlet boundary conditions for the matter fields Θ^α. The operators $\{\mathcal{O}_{2k+1}\}$ are the matter primaries, appropriately dressed by the Liouville field,

$$\mathcal{O}_k = e^{2(1-k)b\phi}\, \mathcal{P}_k(\partial\Theta^\alpha, \bar{\partial}\Theta^\beta)\,. \tag{3.7}$$

To write this expression, we are using the information that the set of matter primaries

$\{\mathcal{P}_k(\partial\Theta^\alpha, \bar\partial\Theta^\beta)\}$ of the Θ^α system have dimensions $\left(\frac{k(k+1)}{2}, \frac{k(k+1)}{2}\right)$. Their explicit expressions can be found in [60].[13] The value of the Liouville dressing follows as usual by requiring that the total dimension be $(1, 1)$.

Recall also that we are taking the bulk cosmological constant $\mu = 0$. (For $\mu \neq 0$, dimensional analysis would dictate the coefficients c_k to be replaced by functions $c_k(\mu_B^2/\mu)$.) It is immediate to determine the powers of μ_B in (3.6) by conservation of the Liouville momentum. One has to recall that each boundary carries a Liouville momentum $Q/2$, and that each factor of μ_B carries momentum $b/2$. This fixes $x_k = 2k+1$. The normalization coefficients c_k could also be computed with some effort, but we shall ignore this here. Consistency of the contact term algebra (section 5) will be an easier route to fix normalizations. Although this logic seems perfectly satisfactory, it would be nice to have a derivation of the same result using the language of *critical* string theory, treating the Liouville theory as an ordinary CFT, in the same spirit as the argument given for branes in imaginary time [36]. The FZZT boundary state can be written as an integral over the continuum spectrum of Liouville momenta $\frac{Q}{2} + iP$ of appropriate Ishibashi states,

$$|\mathrm{FZZT}(\mu_B)\rangle = \int_0^\infty dP\ \Psi(\mu_B, P)\,|\frac{Q}{2} + iP\rangle\,. \tag{3.8}$$

It is conceivable that the analyticity properties of the theory in the complex P plane may allow a contour deformation that would pick up only the poles corresponding to on-shell states in $\frac{b_0}{L_0}\,(|\mathrm{FZZT}(\mu_B)\rangle \otimes |\mathrm{matter}\rangle \otimes |\mathrm{ghost}\rangle)$. This should reduce the boundary state to the same sum of on-shell closed string insertions expected from the quantum gravity argument.

3.2 Boundary CFT

The next logical step is to determine the spectrum of open strings living on these stable branes.

In the open sector of the Θ^α system with Dirichlet boundary conditions, chiral and antichiral oscillators d_n and $\bar d_n$ are identified, and we find a single copy of the chiral current $\partial\Theta^1$ (the same for $\partial\Theta^2$) without any zero modes.[14] It is amusing to check this statement by a modular transformation of the annulus partition function. For this purpose we write the boundary state,

$$\left|\mathcal{B}_\Theta^{\mathrm{Dirichlet}}\right\rangle = \exp\left(\sum_{n=1}^\infty \frac{1}{n}\,\epsilon_{\alpha\beta} d^\alpha_{-n} \bar d^\beta_{-n}\right)\theta_0^1\theta_0^2\,|0\rangle\,. \tag{3.9}$$

The annulus amplitude can be swiftly evaluated,

$$\left\langle \mathcal{B}_\Theta^{\mathrm{Dirichlet}}\right| q^{L_0+1/12}\,\bar q^{\bar L_0+1/12}\left|\mathcal{B}_\Theta^{\mathrm{Dirichlet}}\right\rangle = 2\pi\tilde t\,\eta(\tilde t)^2\,, \qquad q\bar q \equiv e^{-2\pi\tilde t}\,. \tag{3.10}$$

[13] There is in fact a whole SL(2) multiplet of primaries of dimension $\frac{k(k+1)}{2}$ in each chiral half of the theory. However the Θ^α boundary state is an SL(2) singlet (see (3.9)), and this fixes uniquely $\mathcal{P}_k(\partial\Theta^\alpha, \bar\partial\Theta^\beta)$ for each k, as remarked in section 2.1.

[14] Had we defined the $(2, 1)$ string theory using a $\xi\eta$ system, a zero mode for ξ would survive on the boundary ($\xi_0 \equiv \bar\xi_0$, but *one* zero mode is still there). This would spoil our construction.

Modular transformation gives $\eta(t)^2$, which is indeed the same result obtained by tracing over the open string spectrum described above,

$$\mathrm{Tr}_{\mathrm{close}}\left[(-1)^F\, e^{-2\pi t(L_0+1/12)}\right] = \eta^2(t)\,. \tag{3.11}$$

The open string spectrum of the Liouville BCFT for FZZT boundary conditions is known to have the usual primaries $\{e^{\alpha\phi}\}$, of dimension $h_\alpha = \alpha(Q-\alpha)$ (note the factor of two difference with respect to the bulk primaries (2.12)). As usual in Liouville field theory, the continuum spectrum $\alpha = Q/2 + iP$ corresponds to delta-function normalizable states, while real exponents $\alpha \leq Q/2$ correspond to local operators and are used in the dressing of the matter primaries.

A crucial observation, due to Distler [54], is that Liouville and $c = -2$ matter can be formally combined into a $\beta\gamma$ bosonic ghost system of conformal dimensions $(2, -1)$,

$$\beta = \partial\Theta^1 e^{b\phi}\,, \qquad \gamma = \partial\Theta^2 e^{-b\phi}\,. \tag{3.12}$$

(Recall that for $c_{\mathrm{Liou}} = 28$ the parameter $b \equiv 1/\sqrt{2}$). Distler applied this construction to each chiral half of the closed theory, where the Liouville CFT was taken to be a free linear dilaton ($\mu = 0$). The validity of the bosonization formulas (3.12) is then a simple consequence of the free OPEs. This commuting $\beta\gamma$ system has conformal dimensions $(2, -1)$, the same dimensions of the usual anticommuting bc ghost system. This makes the topological nature of the theory intuitively clear. In any open string vacuum amplitude, the oscillator parts of the bc and $\beta\gamma$ path-integrals will exactly cancel each other, and we should expect the only surviving contributions to arise from classical configurations. This expectation will be made more precise below. A basic ingredient is the scalar supersymmetry, or topological charge,

$$Q_S \equiv \oint J_S(z)\,, \qquad J_S(z) \equiv b(z)\gamma(z) = \oint b(z)\partial\Theta^2(z)e^{-b\phi(z)}\,, \tag{3.13}$$

which obeys

$$Q_S^2 = 0\,. \tag{3.14}$$

The usual BRST operator of the bosonic string theory,

$$Q_B = \oint c(z)\left(T^{\mathrm{matter}}(z) + T^{\mathrm{Liou}}(z) + \frac{1}{2}T^{\mathrm{ghost}}(z)\right), \tag{3.15}$$

turns out to be Q_S-exact,

$$Q_B = \left\{Q_S, \oint \frac{1}{2}\beta(z)c(z)\partial c(z)\right\}. \tag{3.16}$$

Turning on the bulk Liouville interaction ($\mu \neq 0$) is expected to preserve the topological nature of the theory, since the Liouville term is Q_S-closed. Here we keep $\mu = 0$ and leave a discussion of the more general case $\mu \neq 0$ to section 6 of the paper.

Crucially for our purposes, an FZZT brane with Dirichlet b.c. for the Θ^α will preserve the total charge $Q_S + \bar{Q}_S$. This is obvious for zero boundary cosmological constant, and holds also for $\mu_B \neq 0$ since the boundary interaction is killed by Q_S^{boundary}. Here we are defining an operator Q_S^{boundary} acting on boundary vertex operators by integrating the current $J_S + \bar{J}_S$ on a semicircle around the boundary operator.

We devote the rest of this section to the computation of the cohomology of Q_S^{boundary}, a technical ingredient that we shall need in our analysis of the open string field theory. There is a slight complication due to the fact that for non-zero boundary cosmological constant μ_B, the BCFT is interacting and the action of Q_S^{boundary} is non-trivial.

Let us first consider the case $\mu_B = 0$. Then the action of Q_S^{boundary} is just the same as for the chiral Q_S operator (3.13) and the cohomology may be readily evaluated. The task is simplified by the realization that the cohomology must lie in the kernel of L_0 and of J_0, the zero-mode of an appropriately defined current $J(z)$. Consider the current[15]

$$J(z) \equiv J_{\text{Liou}}(z) - J_{bc}(z) = \frac{1}{b}\partial\phi + : b(z)c(z) : \,. \tag{3.17}$$

J_{Liou} is an anomalous current that counts the Liouville momentum in units of b, for example $e^{b\phi}$ has J_0 charge one. The linear combination $J(z)$ is non-anomalous and it is Q_S-exact,

$$J(z) = \{Q_S, c(z)\beta(z)\}\,. \tag{3.18}$$

This implies that the cohomology of Q_S is contained in the kernel of J_0. Indeed Q_S is invertible outside this kernel. Similarly, the total energy momentum tensor is Q_S exact. Indeed using (3.16)

$$T(z) = \{Q_B, b(z)\} = \{Q_S, G(z)\}\,, \qquad G(z) \equiv 2\beta(z)\partial c(z) - \partial\beta(z)c(z)\,. \tag{3.19}$$

Hence the cohomology of Q_S is in the kernel of L_0. These two facts readily allow to identify the cohomology of Q_S as the states

$$e^{nb\phi(0)}c(0)\partial c(0)\cdots\partial^n c(0)|0\rangle\,, \qquad e^{-nb\phi(0)}b(0)\partial b(0)\cdots\partial^n b(0)|0\rangle\,. \tag{3.20}$$

When we turn on μ_B the BCFT becomes interacting and the action of Q_S^{boundary} more complicated. Luckily the operator $e^{-b\phi(z)}$ that appears in Q_S is a degenerate field of level two for the Liouville CFT, and its OPEs truncate to two terms,

$$[e^{-b\phi}]\,[e^{\alpha\phi}] = [e^{(\alpha-b)\phi}] + C_-[e^{(\alpha+b)\phi}]\,. \tag{3.21}$$

Hence we can write

$$Q_S^{\text{boundary}} = Q_S^{(0)} + \mu_B^2 Q_S^{(2)}\,. \tag{3.22}$$

Note that for $\mu_B \neq 0$, Q_S^{boundary} does not have definite J_0 charge, but it is a sum of the original charge zero term $Q_S^{(0)}$ plus a deformation of charge two $Q_S^{(2)}$. ($Q_S^{(2)}$ has charge two under J_0 because it has ghost number minus one and shifts the Liouville momentum of $+b$). This is a mild deformation of $Q_S^{(0)}$. Nihilpotency of the total Q_S^{boundary} for any μ_B implies

$$(Q_S^{(0)})^2 = 0 \qquad \{Q_S^{(2)}, Q_S^{(0)}\} = 0\,, (Q_S^{(2)})^2 = 0\,. \tag{3.23}$$

As the J_0 charge of $Q_S^{(2)}$ is nonzero, this implies that $Q_S^{(2)} = \{Q_S^{(0)}, \cdots\}$ and hence it acts trivially on Q_S^0 cohomology.

[15]No confusion should arise between the parameter $b \equiv 1/\sqrt{2}$ and the antighost field $b(z)$!

We conclude that the cohomology of $Q_S^{\text{boundary}} = Q_S^{(0)} + \mu_B^2 Q_S^{(2)}$ has the same dimensionality as the one of $Q_S^{(0)}$: one operator for each ghost number. We will mainly be interested in the ghost number one operator, the open string "tachyon" $e^{b\phi(0)} c_1|0\rangle$. It is immediate to check that this state is in the cohomology for any μ_B. We can repeat the same reasoning also to the BCFT with different boundary cosmological constants μ_B^i and μ_B^j at the two endpoints of the open string. The only states of ghost number one in the cohomology of Q_S^{boundary} are the open tachyons between brane i and brane j,

$$e^{b\phi(0)} c_1|0\rangle_{ij} \,. \tag{3.24}$$

4. Open string field theory and the Kontsevich model

It is our prejudice that open string field theory (OSFT) [26] must play a fundamental role in the understanding of open/closed duality. The Kontsevich model provides the prototypical example. In this section we construct the OSFT on N of the stable branes of the $(2,1)$ string theory, and show how it reduces to the Kontsevich matrix integral.

4.1 Generalities

The OSFT on N D-branes takes quite generally the familiar form

$$S[\Psi] = -\frac{1}{g_s}\left(\frac{1}{2}\sum_{ij}\langle\Psi_{ij}, Q_B\Psi_{ji}\rangle + \frac{1}{3}\sum_{ijk}\langle\Psi_{ij},\Psi_{jk},\Psi_{ki}\rangle\right). \tag{4.1}$$

Let us briefly review the basic ingredients of this action, referring to [72] for background material. The string field $|\Psi_{ij}\rangle$, $i,j = 1,\cdots N$, is an element of the open string state space $\mathcal{H}_{ij}$ between D-brane i and D-brane j. This is the full state-space of the matter + Liouville + ghost BCFT. In classical OSFT, we restrict $|\Psi_{ij}\rangle$ to have ghost number one (in the convention that the $\mathrm{SL}(2,\mathbb{R})$ vacuum $|0\rangle$ has ghost number zero). In the BCFT language, which is the most natural for our purposes, one uses the state-operator map to represent string fields as boundary vertex operators. The string field $|\Psi_{ij}\rangle$ can be expanded as a sum over a complete set of vertex operators,

$$|\Psi_{ij}\rangle = \sum_\alpha c_\alpha \mathcal{V}^\alpha_{ij}(0)|0\rangle \,. \tag{4.2}$$

Here $\mathcal{V}^\alpha_{ij}(0)$ is a vertex operator inserted at the origin of the upper half plane, with boundary conditions for brane i on the negative real axis, and boundary conditions for brane j on the positive real axis.

The 2-point and 3-point vertices are then defined in terms of BCFT correlators on the boundary (real axis) of the upper half-plane,

$$\begin{aligned}\langle A,B\rangle &\equiv \langle I\circ A(0)\, B(0)\rangle_{\text{UHP}}\,, \qquad I(z)\equiv -\frac{1}{z}\\ \langle A,B,C\rangle &\equiv \langle f_1\circ A(0)\, f_2\circ B(0)\, f_3\circ C(0)\rangle_{\text{UHP}}\,.\end{aligned} \tag{4.3}$$

Here $f \circ A(0)$ denotes the conformal transform of the operator $A(0)$ by the complex map f. The precise form of the maps $f_i(z)$, which implement the midpoint gluing of the three open strings, can be found in many places and will not be important for us.

We also recall that the string field obeys the reality condition

$$|\Psi_{ij}\rangle^* = |\Psi_{ji}\rangle\,, \tag{4.4}$$

where the $*$ involution is defined to be [73]

$$* = \mathrm{bpz}^{-1} \circ \mathrm{hc} = \mathrm{hc}^{-1} \circ \mathrm{bpz}\,. \tag{4.5}$$

The operation 'hc' is hermitian conjugation of the state (it sends bras into a kets, with complex conjugation of the coefficients). The operation 'bpz' sends a bra into a ket according to the rule

$$\mathrm{bpz}(\mathcal{V}(0)|0\rangle) = \langle 0| I \circ \mathcal{V}(0)\,. \tag{4.6}$$

Definition of the quantum theory requires gauge-fixing. This is customarily accomplished by imposing Siegel gauge $b_0|\Psi\rangle = 0$. One must introduce Fadeev-Popov ghosts for this gauge fixing, and in fact, since the gauge symmetry is reducible, one needs ghosts for ghosts, and ghosts for ghosts for ghosts, ad infinitum. It is a famous miracle [74] that the full second-quantized gauge-fixed action + ghosts can be written in the form

$$S_{\mathrm{Siegel}} = -\frac{1}{g_s}\left(\frac{1}{2}\sum_{ij}\langle \Psi_{ij}, c_0 L_0 \Psi_{ji}\rangle + \frac{1}{3}\sum_{ijk}\langle \Psi_{ij}, \Psi_{jk}, \Psi_{ki}\rangle\right)\,, \tag{4.7}$$

where $|\Psi_{ij}\rangle$ is now a string field of *unrestricted* ghost number, obeying

$$b_0|\Psi_{ij}\rangle = 0\,. \tag{4.8}$$

The propagator

$$\frac{b_0}{L_0} = \int_0^\infty b_0\, dt\, e^{-t\,L_0} \tag{4.9}$$

has the geometric interpretation of building worldsheet strips of canonical width π and length t. The Feynman diagrams are fatgraphs built joining these flat strips at trivalent vertices (with the curvature concentrated at the common midpoint of the three open strings). This gives the famous decomposition of the moduli space of open Riemann surfaces [18, 47 – 50] which plays a crucial role in Kontsevich construction as well.

4.2 Topological localization

The general OSFT action (4.7) is a very complicated object. In the critical bosonic string, explicit calculations are available for some simple perturbative amplitudes. Off-shell, nonperturbative calculations in the classical theory have so far been possible only using numerical methods (level truncation). In the present case, a drastic simplification occurs thanks to a mechanism of topological localization. A precedent of this phenomenon was discovered by Witten for the topological open A-model on the cotangent bundle $T^*(M)$, which reduces to Chern-Simons on the three-dimensional manifold M.

The localization works in the way familiar for topological theories of cohomological type. The nilpotent supersymmetry $Q_S^{\rm boundary}$ (henceforth simply Q_S) induces a pairing of the states of the theory, such that in a vacuum amplitudes almost all states cancel pairwise; only unpaired states (the cohomology of Q_S) give a non-zero contribution. Let us demonstrate this in a more formal way. We are going to prove that Q_S is a symmetry of the gauge-fixed OSFT action (4.7); moreover the action is almost entirely Q_S-exact, except for the terms involving only the open string tachyons between the N branes. This reduces the OSFT action to an $N \times N$ matrix integral.

The topological symmetry is defined as

$$\delta_S|\Psi\rangle = Q_S|\Psi\rangle\,, \tag{4.10}$$

and it is an invariance of the gauge-fixed action.

$$\delta_S\, S_{\rm Siegel} = 0\,. \tag{4.11}$$

The formal properties that ensure this invariance are

$$\begin{aligned} \langle V_2|\left(Q_S^{(1)}+Q_S^{(2)}\right) &= 0 \\ \langle V_3|\left(Q_S^{(1)}+Q_S^{(2)}+Q_S^{(3)}\right) &= 0\,. \end{aligned} \tag{4.12}$$

Here we are regarding the 2-point and 3-point vertices as elements of $\mathcal{H}^*\otimes\mathcal{H}^*$ and $\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$, i.e., as bilinear and trilinear functionals on the state space $\mathcal{H} = \oplus_{ij}\mathcal{H}_{ij}$. These properties are an immediate consequence of the fact that Q_S is the zero-mode of a conserved current. They are easily proved by contour deformations on the 2- and 3-punctured disks that define the vertices (see e.g. [70]).

We can now apply the general formal arguments given in section 5 of [71] to conclude that the path-integral localizes over the fixed locus of Q_S, that is, over the subspace of states in the cohomology of Q_S. A more lengthy derivation is as follows. We can write

$$\langle V_2| = \langle V_2|^{Q_S\,\rm coho} + \langle W_2|\left(Q_S^{(1)}+Q_S^{(2)}\right), \tag{4.13}$$

$$\langle V_3| = \langle V_3|^{Q_S\,\rm coho} + \langle W_3|\left(Q_S^{(1)}+Q_S^{(2)}+Q_S^{(3)}\right). \tag{4.14}$$

Here we have defined a cohomology problem for Q_S in the spaces $\mathcal{H}^*\otimes\mathcal{H}^*$ and $\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$ in the natural way. Eq. (4.13) is simply the statement that since the 2-point and 3-point vertices are Q_S closed (4.12), they can be written as a sum of a term in the Q_S cohomology plus a Q_S-exact term. By Künneth formula the cohomology in the tensor product space is the tensor product of the cohomology. Thus, dropping Q_S-exact terms, we can restrict the whole OSFT action to the string fields in the cohomology of Q_S.

The cohomology of Q_S was computed in section 3.2 and consists of the states

$$e^{nb\phi(0)}c(0)\partial c(0)\cdots\partial^n c(0)|0\rangle_{ij}\,, \qquad e^{-nb\phi(0)}b(0)\partial b(0)\cdots\partial^n b(0)|0\rangle_{ij}\,. \tag{4.15}$$

Of these states, only the ones with bc ghost number ≥ 1 satisfy the Siegel gauge condition. Among them, only the open string "tachyons"

$$|T_{ij}\rangle \equiv e^{b\phi}c_1|0\rangle_{ij} \tag{4.16}$$

can give a contribution to the action, since all the other fields do not saturate the conservation of bc ghost number, which must add up to three. This concludes the argument that the OSFT action reduces to the terms containing only the open string tachyons.

4.3 Liouville BCFT and the matrix model

Writing the string field $|\Psi_{ij}\rangle$ as

$$|\Psi_{ij}\rangle = X_{ij}|T_{ij}\rangle + \cdots \tag{4.17}$$

for some coefficient X_{ij}, $i,j = 1,\cdots N$, the OSFT reduces to a matrix model for the $N \times N$ matrix X. The reality condition (4.4) for the string field implies that X is hermitian. The action for the matrix integral is

$$S[X] = -\frac{\text{Volume}}{g_s}\left(\frac{1}{2}\,X_{ji}X_{ij}\,\langle T_{ji}, c_0L_0T_{ij}\rangle + \frac{1}{3}\,X_{ij}X_{jk}X_{ki}\,\langle T_{ij},T_{jk},T_{ki}\rangle\right). \tag{4.18}$$

Here we are normalizing the inner products so that

$$\langle c_1, c_0c_1\rangle = 1\,, \tag{4.19}$$

and correspondingly we have extracted a factor of the (divergent) volume of the brane coming from the integration over the zero mode of the Liouville field.[16] It only remains to evaluate the 2- and 3-point vertices for the open string tachyons, which define the coefficients in this matrix action.

The structure of the result can be understood by a simple reasoning. It turns out that for the specific values of Liouville momenta that we are interested in, the effect of μ_B can be treated perturbatively. The Liouville anomaly on the disk is $Q = 3b$. A correlator in which the total Liouville momentum adds to three (in units of b) should then not get any correction from the presence of a boundary cosmological constant. Since the open string tachyon has Liouville momentum one, we expect that the cubic vertex can be evaluated as a free BCFT correlator,

$$\langle T_{ij}, T_{jk}, T_{ki}\rangle = 1\,. \tag{4.20}$$

Notice that the local coordinates $f_i(z)$ play no role since these are on-shell primary vertex operators. On the other hand, in the kinetic term we expect to need one insertion of the boundary cosmological constant to saturate the anomaly. This contribution can come from either side of the strip, so it is reasonable to guess

$$\langle T_{ij}, c_0L_0T_{ij}\rangle \sim \mu_B^{(i)} + \mu_B^{(j)}\,. \tag{4.21}$$

With these values for the coefficients the OSFT action would then become

$$S[X] = -\frac{1}{g_s}\left(\frac{1}{2}X_{ij}X_{ji}(\mu_B^{(i)} + \mu_B^{(j)}) + \frac{1}{3}X_{ij}X_{jk}X_{ki}\right) \tag{4.22}$$

This is the Kontsevich model (1.6), after the identification $\mu_B^{(i)} \equiv z_i$.

[16] This overall factor is present also in all the closed string correlation functions of the the $\mathcal{O}_k$ operators, and it will consistenly cancel out in all formulas.

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One may raise an immediate objection to this reasoning: the kinetic term should actually be zero, since the open tachyon has conformal dimension zero and is thus apparently killed by L_0. Exactly at $c_{\rm Liou} = 28$ there is a loophole in this objection, because the scalar product $\langle T_{ij}, c_0 T_{ji}\rangle$ is divergent. A more careful analysis is then called for, involving the full machinery of Liouville BCFT.

To regulate the divergence in the tachyon 2-point function, we can go slightly off-shell, considering the state $e^{(b+\epsilon)\phi}c_1|0\rangle_{ij}$. As we show in the appendix, the 2-point function in boundary (FZZT) Liouville theory has a pole as $\epsilon \to 0$, precisely with the expected residue,

$$\langle e^{(b+\epsilon)\phi} e^{(b+\epsilon)\phi}\rangle_{1,2} \sim \frac{\mu_B^{(1)} + \mu_B^{(2)}}{\epsilon}\,. \tag{4.23}$$

This pole cancels the zero from the action of L_0,

$$L_0\, e^{\alpha\phi}c_1|0\rangle = (\alpha - b)(\alpha - 2b)e^{\alpha\phi}c_1|0\rangle = \epsilon\,(-b)\,e^{\alpha\phi}c_1|0\rangle\,, \tag{4.24}$$

giving the desired result. The careful computation of the 3-point function (see the appendix) is rather uneventful and confirms (4.20). This resonant behavior of Liouville field theory correlators is related to the fact that the critical exponent $\gamma_{str} \equiv 1 - 1/b^2$ equals minus one. In general, a similar resonant behavior occurs when γ_{str} is a negative integer [56]. The corresponding values of the central charge $c_{\rm Liou} = 1 + 6(p+1)^2/p$, with integer $p \geq 2$, are precisely the ones needed to dress the matter minimal models $(p, 1)$. These are also the models where the string theory is known to be topological and a matrix model *à la* Kontsevich exists.

4.4 Discussion

We have seen that only on-shell fields (the open string tachyons) give non-zero contributions. This can be given a geometric interpretation: the whole vacuum amplitude has support on the region of moduli space where all propagator lengths in the fatgraph diverge. The localization on such singular Riemann surfaces is again familiar from the Chern-Simons example [27]. In the language of [27], we can say that there are no ordinary instantons, and only virtual instantons at infinity contribute. It is well-known that in topological gravity closed string amplitudes are localized on singular surfaces [22, 46]. Here we are seeing this phenomenon in the open channel. While in the closed channel contact terms are quite intricate, the open string moduli space is structurally much simpler, and open string contact terms arise only when boundaries touch each other or pinch. This geometric intuition could be used to streamline the combinatorial proofs [75, 76] of the Virasoro constraints for the Kontsevich model.

5. Open/closed duality and Ward identities

The main conclusion to draw is that in this theory, the effect of D-branes can be completely accounted for by turning on a simple source term for the closed strings,

$$\mathcal{Z}^{\rm close}(g_s, \{z_i\}) = \mathcal{Z}^{\rm closed}\left(g_s, \left\{t_k = g_s \sum_i \frac{1}{k\, z_i^k}\right\}\right). \tag{5.1}$$

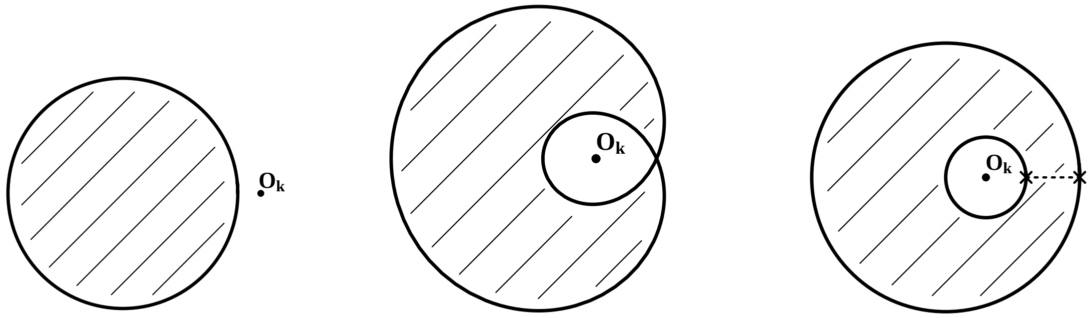

Figure 3: Degeneration of the Riemann surface as the closed string operator $\mathcal{O}_k$ approaches the boundary. The shadowed region represents the hole. As the short neck pinches, the surface factorizes into two surfaces, each with the extra insertion of an open string tachyon, indicated by a cross.

This conclusion can be strengthened by considering the partition function of the theory in the presence of both a D-brane and a non-trivial closed string background.[17] Recall that in the closed string theory, the partition function is completely determined by the Virasoro Ward identities [46, 80]

$$\begin{aligned}
\frac{\partial}{\partial t_1}\mathcal{Z} = \mathcal{L}_{-2}\mathcal{Z} &\equiv \frac{t_1^2}{2g_s^2}\mathcal{Z} + \sum_{k=0}^{\infty}(2k+3)t_{2k+3}\frac{\partial \mathcal{Z}}{\partial t_{2k+1}} \\
\frac{\partial}{\partial t_3}\mathcal{Z} = \mathcal{L}_{0}\mathcal{Z} &\equiv \frac{1}{8}\mathcal{Z} + \sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2k+1}} \\
\frac{\partial}{\partial t_{2n+5}}\mathcal{Z} = \mathcal{L}_{2n+2}\mathcal{Z} &\equiv \sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2k+2n+1}} + \frac{g_s^2}{2}\sum_{k=0}^{n}\frac{\partial^2 \mathcal{Z}}{\partial t_{2k+1}\partial t_{2n-2k+1}}\,.
\end{aligned} \tag{5.2}$$

Each of these equations details how a specific $\mathcal{O}_k$ operator, when integrated over the Riemann surface, picks contributions from collision with other operators or with nodes of the surface [46, 80]. The second term in the $\mathcal{L}_{-2}$ and $\mathcal{L}_0$ equations, and the first term in the $\mathcal{L}_{2n+2}$ equation, represent the collision of two operators. The last term in the $\mathcal{L}_{2n+2}$ equation represents the collision between an operator and a node. (The first term in the $\mathcal{L}_{-2}$ equation accounts for the conformal Killing vectors of the sphere, and similarly the first term in the $\mathcal{L}_0$ equation accounts for the CKV of the torus.) The structure of these equations is strongly constrained by self-consistency; it is only because the $\mathcal{L}_{2n}$ form (half) a Virasoro algebra that these equations have a solution. To find the partition function when both D-brane sources and closed string sources are turned on, we will now extend these Ward identities by adding the contact terms that arise from the new ways the surface can degenerate: when an operator $\mathcal{O}_k$ collides with a boundary; and when a boundary collides with a node. The collision of an operator with a boundary has the schematic aspect shown in figure 3.

[17] After submitting the first version of this paper, we learnt that a similar approach as the one outlined in this section was already developed in the "old" days of matrix models in the interesting works [77, 78] (see also the recent paper [79]). We thank C. Johnson for pointing out these references to us.

The short neck of the pinching surface is conformally equivalent to the insertion of a very long open string propagator; the collision leaves behind an open string tachyon insertion, with a power of z fixed by conservation of the Liouville momentum. This piece of knowledge, together with the requirement that we still have a Virasoro algebra, uniquely fixes the open + closed Ward identities. Considering for simplicity the case of a single D-brane with parameter z, they have the following form:

$$\begin{aligned}\frac{\partial}{\partial t_1}\mathcal{Z} &= \tilde{\mathcal{L}}^{(z)}_{-2}\mathcal{Z} \equiv \mathcal{L}_{-2}\mathcal{Z} + \left(\frac{t_1}{z g_s} + \frac{1}{2z^2}\right)\mathcal{Z} - \frac{1}{z}\frac{\partial \mathcal{Z}}{\partial z} \\ \frac{\partial}{\partial t_3}\mathcal{Z} &= \tilde{\mathcal{L}}^{(z)}_{0}\mathcal{Z} \equiv \mathcal{L}_{0}\mathcal{Z} - z\frac{\partial \mathcal{Z}}{\partial z} \\ \frac{\partial}{\partial t_{2n+5}}\mathcal{Z} &= \tilde{\mathcal{L}}^{(z)}_{2n+2}\mathcal{Z} \equiv \mathcal{L}_{2n+2}\mathcal{Z} - z^{2n+1}\frac{\partial \mathcal{Z}}{\partial z} - g_s\sum_{k=0}^{n} z^{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2n-2k+1}}\,. \end{aligned} \tag{5.3}$$

The terms involving $\frac{\partial \mathcal{Z}}{\partial z}$ represent the collision of an operator with a boundary. The last term in the $\tilde{\mathcal{L}}^{(z)}_{2n+2}$ equation represents the collision of a boundary and a node. Finally the second term in the $\tilde{\mathcal{L}}^{(z)}_{-2}$ equation accounts for the CKV of the disk with two closed punctures and of the annulus with one closed puncture.

These identities are sufficient to completely determine the open + closed partition function $\mathcal{Z}^{\text{open+closed}}(g_s,\{t_k\},\{z_i\})$. Not surprisingly, one can easily verify that the solution is

$$\mathcal{Z}^{\text{open+closed}}(g_s,\{t_k\},\{z_i\}) = \mathcal{Z}^{\text{closed}}\left(g_s, \left\{t_k + g_s\sum_i \frac{1}{k\, z_i^k}\right\}\right). \tag{5.4}$$

This shows that even when there are non-trivial closed string sources to begin with, D-branes can still be re-absorbed into a shift of these sources. This argument also fixes the overall normalization in the relation between t_k and $\sum_i z_i^{-k}$. The closed operators $\mathcal{O}_k$ have an intrinsic normalization fixed by the algebra of closed contact terms. The algebra of open/closed contact terms can then be used to fix the coefficients of these canonically normalized $\mathcal{O}_k$ in the expansion of the boundary state. This ties a loose end in our derivation of the Kontsevich model.

We can also define an open partition function in a non-trivial closed background by subtracting the purely closed amplitudes,

$$\mathcal{Z}^{\text{close}}(g_s, z_i\,|\,t_k) = \frac{\mathcal{Z}^{\text{open+closed}}(g_s,t_k,z_i)}{\mathcal{Z}^{\text{closed}}(g_s,t_k)}\,. \tag{5.5}$$

An interesting question is whether this open partition function is computed by an appropriate generalization of the Kontsevich matrix model. In the next section we provide the answer for the background with $\mu = t_1 \neq 0$ and $t_k = 0$, $k > 1$.

6. Non-zero bulk cosmological constant

As shown in the appendix, the matrix model obtained by topological localization of the OSFT action depends only on the boundary cosmological constants $\{\mu_B^i\}$ and not on μ.

Clearly however the open + closed string partition function $\mathcal{Z}^{\text{open+closed}}(g_s, \{t_k = \mu\delta_{1,k}\}\,|z_i)$ has a non-trivial μ dependence. The resolution of this apparent contradiction is that as we turn on $\mu \neq 0$, we must change the dictionary between the open moduli (the values of the boundary cosmological constants $\{\mu_B^i\}$) and the closed moduli $\{t_k\}$. Indeed, the relation used so far is based on the expansion (3.6) of the boundary state, which - as written - is valid only for $\mu = 0$.

As we turn on $\mu \neq 0$, we use conventions where the parameters $\{z_i\}$ are related to the sources $\{t_k\}$ just as before,

$$t_k = \frac{g_s}{k}\sum_{n=1}^{N}\frac{1}{z_k^n}\,, \tag{6.1}$$

but we do not identify anymore z with μ_B, rather $z = f(\mu, \mu_B)$ for some function f which we now proceed to determine.

To this end, we use the Ward identities derived in the previous section. The free energy

$$\mathcal{F}(g_s, \mu, \{z_i\}) \equiv \log(\mathcal{Z}^{\text{open+closed}}(g_s, \{t_k = \mu\delta_{1,k}\}\,|\,\{z_i\})) \tag{6.2}$$

satisfies

$$\frac{\partial}{\partial\mu}\mathcal{F} + \sum_i \frac{1}{z_i}\frac{\partial \mathcal{F}}{\partial z_i} = \frac{\mu^2}{2g_s^2} + \sum_i \frac{\mu}{z_i g_s} + \sum_{i,j}\frac{1}{2z_i z_j}\,. \tag{6.3}$$

This equation can be readily integrated. One finds

$$\begin{aligned}\mathcal{F}(g_s, \mu, z_i) = &\frac{\mu^3}{6g_s^2} + \sum_i \frac{1}{g_s}\left[\frac{1}{3}(z_i^2 - 2\mu)^{\frac{3}{2}} - \frac{z_i^3}{3} + \mu z_i\right] + \frac{1}{2}\sum_{i,j}\log\frac{z_i + z_j}{(z_i^2 - 2\mu)^{\frac{1}{2}} + (z_j^2 - 2\mu)^{\frac{1}{2}}} + \\ &+\mathcal{F}\left(g_s, 0, (z_i^2 - 2\mu)^{\frac{1}{2}}\right).\end{aligned} \tag{6.4}$$

The first three terms in this expression represent respectively the changes of the sphere, disk and annulus amplitudes as we turn on μ. The last term is the sum of all vacuum diagrams with at least two holes, given as usual by the Kontsevich matrix integral (1.6), but with the replacement $z \to (z^2 - 2\mu)^{\frac{1}{2}}$. From the analysis in the appendix we know that the kinetic term in the Kontsevich integral is to be identified with the boundary cosmological constant even for $\mu \neq 0$, hence we learn $\mu_B = (z^2 - 2\mu)^{\frac{1}{2}}$, which gives the sought relation

$$z = (\mu_B^2 + 2\mu)^{\frac{1}{2}}\,. \tag{6.5}$$

So far we have argued that consistency of the theory demands this new relation between open and closed moduli, which is forced upon us by the open/closed integrable structure. Now we wish to give an independent check of this logic, and in the process obtain a more physical interpretation.

We start with the Kontsevich representation of the partition function,

$$\begin{aligned}\mathcal{Z}(g_s, 0, (z_i^2 - 2\mu)^{\frac{1}{2}}) &\equiv \exp\left(\mathcal{F}\left(g_s, 0, (z_i^2 - 2\mu)^{\frac{1}{2}}\right)\right) \\ &= \rho\Big((Z^2 - 2\mu)^{\frac{1}{2}}\Big)^{-1}\int [dX]\,\exp\left(\frac{1}{g_s}\,\mathrm{Tr}\left[-\frac{1}{2}(Z^2 - 2\mu)^{\frac{1}{2}}X^2 + \frac{1}{6}X^3\right]\right),\end{aligned} \tag{6.6}$$

and perform the shift $X \to X + (Z^2 - 2\mu)^{\frac{1}{2}} - Z$ in the integration variable. This gives

$$\mathcal{Z}\left(g_s, 0, (z_i^2 - 2\mu)^{\frac{1}{2}}\right) = \exp(-\mathcal{F}^{D^2}) \cdot \rho\left((Z^2 - 2\mu)^{\frac{1}{2}}\right)^{-1} \times \\ \times \int [dX] \exp\left(\frac{1}{g_s} \operatorname{Tr}\left[-\frac{1}{2} Z X^2 + \frac{1}{6} X^3 + \mu X\right]\right), \qquad (6.7)$$

where $\mathcal{F}^{D^2} \equiv \left[\frac{1}{3}(z_i^2 - 2\mu)^{\frac{3}{2}} - \frac{z_i^3}{3} + \mu z_i\right]$ is exactly the second term in (6.4). We observe that all the terms conspire to give a simple expression for the full open/closed partition function,[18]

$$\mathcal{Z}(g_s, \mu, z_i) = \exp\left(\frac{\mu^3}{6 g_s^2}\right) \cdot \rho(Z)^{-1} \int [dX] \exp\left(\frac{1}{g_s} \operatorname{Tr}\left[-\frac{1}{2} Z X^2 + \frac{1}{6} X^3 + \mu X\right]\right). \qquad (6.8)$$

This final equation has a transparent interpretation. Apart from the purely closed contribution $\exp\left(\mu^3/6g_s^2\right)$ coming from the sphere, the partition function is computed by the OSFT in the trivial background $\mu = 0$ (i.e., with the usual kinetic term), but with the addition on an extra linear term μX.

This is precisely what we would expect if the effect of deforming the closed string background to $\mu \neq 0$ was captured by an open/closed vertex linear in the open string field. In fact, it is well-known that OSFT can reproduce amplitudes with closed string insertions and at least a boundary by adding to the action an appropriate open-closed vertex [81], a linear term coupling the closed string vertex operators to the open string fields. Since the cosmological constant operator is Q_S-closed, the open-closed vertex does not ruin the topological localization, and reduces exactly to $\frac{\mu}{g_s} \operatorname{Tr} X$ in the matrix integral! In more complicated string theories, we would not expect in general to be able to exponentiate a finite defomation of the closed string background by simply adding this linear term, but evidently this procedure is justified here. In particular cosmological constant operator $\mathcal{O}_1$ does not have contact terms with itself which would obstruct a naive exponentiation.

We see here what may well be the simplest illustration of background independence in string field theory. We can either start from the trivial background $\mu = 0$ and shift μ through the open/closed vertex, as in (6.8), or formulate directly the theory in the new background with $\mu \neq 0$, as in (6.6). Background indipendence dictates that the two forms of the action must be related by a field redefinition, which in this case is just a linear shift of the "string field" X. Is it possible to turn on other sources t_k using the same procedure? For t_3, corresponding to the dilaton operator $\mathcal{O}_3$, the Ward identity can be integrated in a similar way and it simply gives an appropriate rescaling of the relation between μ_B and z. This is equivalently expressed by adding to the matrix action the simple open-closed vertex $\frac{3t_3}{g_s} \operatorname{Tr} Z^2 X$, just as expected. This procedure is not expected to work as easily for higher t_k's, as the operators now have a non-trivial algebra of contact terms. Rather one may anticipate a complicated matrix action containing multi-trace interactions.

[18]Notice that the third term in (6.4) is precisely taken into account by the ρ prefactors.

Finally we should briefly outline how the analysis of this section could be recast in the language of integrable hierarchies. Turning on μ corresponds to moving in the "small phase space" (which for the (2,1) model contains only the operator $\mathcal{O}_1$). The relation between the KP times t_k and the coordinates $\{z_i\}$ changes according to well-known formulas (see e.g. sections 4.2–4.3 of [40]) which could have been used to deduce the relation (6.5). Here we have phrased the discussion in a perhaps more intuitive physical language.

7. Future directions

There are many interesting directions in which the work of this paper may be continued. In this section we mention some of them.

7.1 Relation with discretized random surface in $D = -2$

In this paper we have focused on the Kontsevich model for the $(2,1)$ string theory. There is also a double-scaled matrix model for this closed string theory, defined in terms of a matrix $M(\theta^1, \theta^2)$ that depends of two Grassmann-odd coordinates [55 – 59]. This model has a rich structure with many intriguing properties.

In the continuum limit, the coordinates θ^1 and θ^2 become precisely our fields Θ^α. This is one of the reasons why one should prefer the Θ^α system to the $\xi\eta$ system. Following the philosophy of [7], this doubled-scaled matrix model should be understood as the open string field theory on *unstable* D-branes of the theory. Indeed, if one considers in the continuum $(2,1)$ string theory ZZ boundary conditions for the Liouville direction, and Neumann b.c. for the Θ^α system, one finds that the tachyon dynamics is captured by a matrix $M(\theta_0^1, \theta_0^2)$, where θ_0^α are the zero-modes of Θ^α living on the Neumann boundary.

In [58], macroscopic loop operators for this matrix model are considered. The operators of topological gravity appear to be related to loop operators with Dirichlet boundary conditions on the θ^α. This seems to agree with our construction, and it would be nice to understand this connection in detail.

More generally, it is of interest to see whether our approach can shed some light on open/closed duality [7] for the double-scaled matrix models. In the "old" approach, the doubled-scaled matrix model is thought of as a trick to discretize the Riemann surface, and it is essential to send N to infinity and $t \to t_c$ to recover the continuum theory. The modern approach starts instead from considering the worldvolume theory of a finite number N of ZZ branes in the continuum string theory. The precise relation between the old and the new approach is still quite unclear, as one cannot directly identify the finite N matrix model before double-scaling limit with the finite N open string field theory of the ZZ branes. The OSFT of N ZZ branes, with N finite, is presumably a unique and consistent continuum quantum theory, while the finite N matrix model has non-universal features, like the precise form of the potential. The OSFT on N ZZ branes may be expected [83] to be dual to a subsector of the full continuum closed string theory. This is in analogy with the finite N Kontsevich model.[19]

[19] We thank Ashoke Sen for pointing out this analogy.

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7.2 Generalizations

The most obvious generalization of this work that comes to mind is to the other (p, q) minimal string theories. (p, q) theories are solved by double-scaling of the $(p-1)$-matrix chain, where again q labels the order of criticality. $(p, 1)$ models represent the "topological points", from which the (p, q) models with $q > 1$ are obtained by flows of the p-KdV hierarchy. There is a Kontsevich model for any $(p, 1)$ theory, it is a one-matrix integral with a potential of order $p+1$. Our logic leads us to believe that the OSFT on the stable branes of the $(p, 1)$ theory will localize topologically to a matrix integral. Since OSFT is cubic, this process will lead to a cubic matrix integral involving several matrices (a matrix for each open topological primary). The simplest guess is that such cubic models are related to the known polynomial Kontsevich models by integrating out all matrices but one. A formulation in terms of a cubic multi-matrix integral may have the advantage of making more transparent the relation with a decomposition of moduli space, which has not been completly understood for the intersection numbers associated to the $(p, 1)$ models. Work is in progress along these lines.

Several other generalizations can be contemplated. $\hat{c} < 1$ theories admit topological points and to the best of our knowledge there is no known topological matrix model description; our procedure should give one. The case of $c = 1$ at the self-dual radius should also be attacked.

8. Conclusions

In this paper we have described an example of exact open/closed duality that should represent the simplest paradigm for a large class of similar dualities. The worldsheet picture of holes shrinking to punctures is not, we believe, an artifact of the simplicity of the model, and the same mechanism may be at work in more physical situations. We have found that at least in this example, open string field theory on an infinite number of branes is capable of describing the full string theory. This may contain a more general lesson.[20] Although here we have stressed the importance of open string field theory as a tool to understand open/closed duality, one of our original motivations was to learn about the structure of OSFT itself in the solvable context of low-dimensional string theories. The Kontsevich model is arguably the simplest imaginable OSFT. It is still a good question whether this and related examples can be used to sharpen our understanding of OSFT. We would like to conclude with a speculation about how open/closed duality may come about in AdS/CFT. The example of the Kontsevich model suggests that the natural starting point is the closed string theory dual to free SYM ('t Hooft parameter $t = 0$). At the point $t = 0$, which in some sense must correspond to an infinitely curved AdS space, the closed string theory is expected to have an infinite dimensional symmetry group. This is analogous to the statement that $\{t_k = 0\}$ is the topological point of the Kontsevich model. If a a concrete description of this closed string theory were available, one may also hope to

[20] Open string field theory on an infinite number of branes has been conjectured [82] to be relevant for the issue of background independence in string theory.

define D-branes. D-branes of a peculiar nature may exist, such that: 1) The open string field theory on these D-branes is precisely the SYM theory, with no extra massive open string modes. 2)When considered in the closed string channel, the presence of the D-brane can be completely re-adsorbed in a shift of the closed string background. Adding D-branes would then be equivalent to turning on a finite t, that is, to recovering a smooth AdS space. Statement 1) is analogous to the topological localization that we have described for the Kontsevich model, while statement 2) is the by now familiar mantra of replacing boundaries with punctures. This scenario would offer a derivation of AdS/CFT orthogonal to the usual one [31] that begins with D-branes in flat space and proceeds by "dropping the one" in the harmonic function.

Acknowledgments

It is a pleasure to thank Sunny Itzhaki, Igor Klebanov, John McGreevy, Nathan Seiberg, Joerg Teschner and especially Ashoke Sen and Herman Verlinde for very useful discussions and comments. This material is based upon work supported by the National Science Foundation Grant No. PHY-0243680. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

A. Liouville BCFT correlators

In this appendix we give the technical details of the computation of 2- and 3-point vertices of open string tachyons.

We need the explicit expressions of 2- and 3-point functions of boundary primary operators in Liouville BCFT (with FZZT boundary conditions). The relevant formulas can be found in [14, 69]. We use the notations of [14]. The variable s is conventionally introduced[21]

$$\frac{\mu_B}{\sqrt{\mu}} = \cosh b\pi s \,. \tag{A.1}$$

Here μ is the bulk cosmological constant. We are interested in the limit $\mu \to 0$, since this is the point $\{t_k = 0\}$. Interestingly, the results for 2- and 3-point correlators of open string tachyon turn out to be independent of μ. An important ingredient is the special function $\mathbf{G}_b(x)$ defined in [14]. This function is entire-analytic and has zeros for $x = -nb - m/b$, with $m, n = 0, 1, 2, \ldots$; it is symmetric under $b \leftrightarrow 1/b$. A convenient combination of $\mathbf{G_b}$'s is the function $\mathbf{S}_b(x) = \mathbf{G}_b(Q-x)/\mathbf{G}_b(x)$, which obeys the shift relation

$$\mathbf{S}_b(x+b) = 2\sin(\pi b x)\mathbf{S}_b(x) \,. \tag{A.2}$$

[21] The FZZT BCFT shows an interesting monodromy in the complex μ_B plane [84]. The physics is instead entire-analytic in terms of s.

The 2-point function of boundary primary fields is then [14]

$$
\begin{aligned}
d(\alpha,\mu_B^{(1)},\mu_B^{(2)},\mu) &\equiv \langle e^{\alpha\phi}e^{\alpha\phi}\rangle \\
&= \left(\frac{\pi}{\sqrt{2}}\mu\gamma\left(\frac{1}{2}\right)\right)^{\frac{3}{2}-\sqrt{2}\alpha}\times \\
&\quad\times\frac{\mathbf{G}_{\frac{1}{\sqrt{2}}}\left(-2\alpha+\frac{3}{\sqrt{2}}\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(\frac{3}{\sqrt{2}}+i\frac{(s_1+s_2)}{2}-\alpha\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(\frac{3}{\sqrt{2}}+i\frac{(s_1-s_2)}{2}-\alpha\right)}{\mathbf{G}_{\frac{1}{\sqrt{2}}}\left(-\frac{3}{\sqrt{2}}+2\alpha\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(i\frac{(s_1+s_2)}{2}+\alpha\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(i\frac{(s_1-s_2)}{2}+\alpha\right)}\,.
\end{aligned}
\tag{A.3}
$$

We now take $\alpha = b+\epsilon$. As $\epsilon\to 0$ there is a pole arising from the zero of the first $\mathbf{G}_b$ in the denominator. The interesting residue is contained in the part of the expression, finite for $\alpha\to b=\frac{1}{\sqrt{2}}$, that contains the four $\mathbf{S}_{\frac{1}{\sqrt{2}}}$ functions,

$$
\begin{aligned}
&\frac{\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(\frac{2}{\sqrt{2}}+i\frac{(s_1+s_2)}{2}\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(\frac{2}{\sqrt{2}}+\frac{(s_1-s_2)}{2}\right)}{\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(i\frac{(s_1+s_2)}{2}+\frac{1}{\sqrt{2}}\right)\mathbf{S}_{\frac{1}{\sqrt{2}}}\left(i\frac{(s_1-s_2)}{2}+\frac{1}{\sqrt{2}}\right)}= \\
&\qquad\qquad = 4\sin\left(\frac{\pi}{2}+\frac{i\pi}{2\sqrt{2}}(s_1+s_2)\right)\sin\left(\frac{\pi}{2}+\frac{i\pi}{2\sqrt{2}}(-s_1+s_2)\right) \\
&\qquad\qquad = 2\cosh\left(\frac{\pi}{\sqrt{2}}s_1\right)+2\cosh\left(\frac{\pi}{\sqrt{2}}s_2\right)=2\frac{\mu_B^{(1)}+\mu_B^{(2)}}{\sqrt{\mu}}\,.
\end{aligned}
\tag{A.4}
$$

The factor of $1/\sqrt{\mu}$ cancels against the $\sqrt{\mu}$ in the prefactor of (A.3). This proves the claim (4.23). The three point function simplifies when one takes the three Liouville momenta to be equal to b. For generic b, this 3-point function is proportional to a rational function of μ, μ_B and the "dual" cosmological constant $\tilde{\mu}_B$ [69],

$$
\langle e^{b\phi}\,e^{b\phi}\,e^{b\phi}\rangle\sim\frac{\tilde{\mu}_B^{(1)}\left(\mu_B^{(2)}-\mu_B^{(3)}\right)+\tilde{\mu}_B^{(2)}\left(\mu_B^{(3)}-\mu_B^{(1)}\right)+\tilde{\mu}_B^{(3)}\left(\mu_B^{(1)}-\mu_B^{(2)}\right)}{\left(\mu_B^{(2)}-\mu_B^{(3)}\right)\left(\mu_B^{(3)}-\mu_B^{(1)}\right)\left(\mu_B^{(1)}-\mu_B^{(2)}\right)}. \tag{A.5}
$$

For $c_{\rm Liou}=28$, the dual cosmological constant obeys

$$
\tilde{\mu}_B^{(i)}\sim\left(2(\mu_B^{(i)})^2-\mu\right) \tag{A.6}
$$

and the tachyon 3-point function is just a constant independent of μ and μ_B^i.

Here we have computed the Liouville correlators using analytic continuation in the Liouville momentum. (Equally well, we could have use analytic continuation in b to regulate the expressions that become singular as $b\to 1/\sqrt{2}$. Indeed one of the achievements of the past few years has been the recognition that Liouville correlators have nice analytic properties with respect to all the parameters.) If one insists in working strictly at $b=1/\sqrt{2}$ and with the on-shell vertex operators $e^{b\phi}$, an alternative way to phrase the results is the language of logarithmic CFT [61]. For generic b, the two operators $e^{\alpha\phi}$ and $e^{(Q-\alpha)\phi}$ are identified as

$$
e^{\alpha\phi}=d\left(\alpha,\mu_B^{(1)},\mu_B^{(2)},\mu\right)e^{(Q-\alpha)\phi}\,. \tag{A.7}
$$

The reflection coefficient $d(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu)$ has poles for $Q - 2\alpha = nb + m/b$. For these cases, the identification becomes ill-defined. One way around this is that for these resonant values $\tilde{\alpha}$ we modify the identification as

$$(L_0 - h_{\tilde{\alpha}})\, e^{\tilde{\alpha}\phi} = \left[\lim_{\alpha\to\tilde{\alpha}} \Big((h_\alpha - h_{\tilde{\alpha}}) d\Big(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu\Big)\Big) \right] e^{(Q-\tilde{\alpha})\phi} . \tag{A.8}$$

Notice that the term is in square brackets is just a finite coefficient. L_0 cannot be diagonalized in the subspace spanned by $e^{\tilde{\alpha}\phi}$ and $e^{(Q-\tilde{\alpha})\phi}$, which forms a non-trivial Jordan cell. In other terms, the two operators are a logarithmic pair. In our case, $\tilde{\alpha} = b$. Working at b strictly equal to $1/\sqrt{2}$, we can write

$$L_0\, e^{\phi(0)/\sqrt{2}} c_1 |0\rangle_{ij} = \Big(\mu_B^{(i)} + \mu_B^{(j)}\Big) e^{\sqrt{2}\phi(0)} c_1 |0\rangle_{ij} . \tag{A.9}$$

This gives an alternative way to understand why the tachyon kinetic term in the OSFT action is $(\mu_B^{(i)} + \mu_B^{(j)})$.

References

[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[3] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2-d gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[4] I.R. Klebanov, *String theory in two-dimensions*, hep-th/9108019.

[5] P.H. Ginsparg and G.W. Moore, *Lectures on 2-d gravity and 2-d string theory*, hep-th/9304011.

[6] J. Polchinski, *What is string theory?*, hep-th/9411028.

[7] J. McGreevy and H.L. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *JHEP* **12** (2003) 054 [hep-th/0304224].

[8] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *JHEP* **07** (2003) 045 [hep-th/0305159].

[9] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[10] T. Takayanagi and N. Toumbas, *A matrix model dual of type 0B string theory in two dimensions*, *JHEP* **07** (2003) 064 [hep-th/0307083].

[11] M.R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, *A new hat for the $c = 1$ matrix model*, hep-th/0307195.

[12] I.R. Klebanov, J. Maldacena and N. Seiberg, *Unitary and complex matrix models as 1-d type 0 strings*, *Commun. Math. Phys.* **252** (2004) 275 [hep-th/0309168].

[13] E.J. Martinec, *The annular report on non-critical string theory*, hep-th/0305148;
N.R. Constable and F. Larsen, *The rolling tachyon as a matrix model*, *JHEP* **06** (2003) 017 [hep-th/0305177];
J. McGreevy, J. Teschner and H.L. Verlinde, *Classical and quantum D-branes in 2d string theory*, *JHEP* **01** (2004) 039 [hep-th/0305194];
V. Schomerus, *Rolling tachyons from Liouville theory*, *JHEP* **11** (2003) 043 [hep-th/0306026];
A. Sen, *Open-closed duality at tree level*, *Phys. Rev. Lett.* **91** (2003) 181601 [hep-th/0306137];
S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *JHEP* **09** (2003) 057 [hep-th/0306177];
D. Gaiotto, N. Itzhaki and L. Rastelli, *On the BCFT description of holes in the $c = 1$ matrix model*, *Phys. Lett.* **B 575** (2003) 111 [hep-th/0307221];
M. Gutperle and P. Kraus, *D-brane dynamics in the $c = 1$ matrix model*, *Phys. Rev.* **D 69** (2004) 066005 [hep-th/0308047];
A. Sen, *Open-closed duality: lessons from matrix model*, *Mod. Phys. Lett.* **A 19** (2004) 841 [hep-th/0308068];
J. McGreevy, S. Murthy and H.L. Verlinde, *Two-dimensional superstrings and the supersymmetric matrix model*, *JHEP* **04** (2004) 015 [hep-th/0308105];
A. Giveon, A. Konechny, A. Pakman and A. Sever, *Type 0 strings in a 2-d black hole*, *JHEP* **10** (2003) 025 [hep-th/0309056];
J.L. Karczmarek and A. Strominger, *Matrix cosmology*, *JHEP* **04** (2004) 055 [hep-th/0309138];
S. Ribault and V. Schomerus, *Branes in the 2-d black hole*, *JHEP* **02** (2004) 019 [hep-th/0310024];
S. Dasgupta and T. Dasgupta, *Renormalization group approach to $c = 1$ matrix model on a circle and D-brane decay*, hep-th/0310106;
S. Alexandrov, *(m, n) ZZ branes and the $c = 1$ matrix model*, *Phys. Lett.* **B 604** (2004) 115 [hep-th/0310135];
J. Gomis and A. Kapustin, *Two-dimensional unoriented strings and matrix models*, *JHEP* **06** (2004) 002 [hep-th/0310195].

[14] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[15] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138; *Liouville theory revisited*, *Class. and Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[16] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[17] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, *Unification of all string models with $c < 1$*, *Phys. Lett.* **B 275** (1992) 311 [hep-th/9111037];
M. Adler and P. van Moerbeke, *A matrix integral solution to two-dimensional $W(P)$ gravity*, *Commun. Math. Phys.* **147** (1992) 25;
L. Bonora and C.S. Xiong, *The (N, M)-th KdV hierarchy and the associated W algebra*, *J. Math. Phys.* **35** (1994) 5781 [hep-th/9311070];
B. Dubrovin, *Geometry of 2-d topological field theories*, hep-th/9407018;
A. Morozov, *Matrix models as integrable systems*, hep-th/9502091;
S. Kharchev, *Kadomtsev-Petviashvili hierarchy and generalized Kontsevich model*, hep-th/9810091.

[18] R. Penner, *Perturbative series and the moduli space of Riemann surfaces*, *J. Diff. Geom.* **27** (1988) 35.

[19] R. Dijkgraaf, G.W. Moore and R. Plesser, *The partition function of 2-d string theory*, *Nucl. Phys.* **B 394** (1993) 356 [hep-th/9208031];
C. Imbimbo and S. Mukhi, *The topological matrix model of $c = 1$ string*, *Nucl. Phys.* **B 449** (1995) 553 [hep-th/9505127].

[20] S.Y. Alexandrov, V.A. Kazakov and I.K. Kostov, *2d string theory as normal matrix model*, *Nucl. Phys.* **B 667** (2003) 90 [hep-th/0302106].

[21] S. Mukhi, *Topological matrix models, Liouville matrix model and $c = 1$ string theory*, hep-th/0310287.

[22] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, *Nucl. Phys.* **B 340** (1990) 281.

[23] R. Dijkgraaf and E. Witten, *Mean field theory, topological field theory and multimatrix models*, *Nucl. Phys.* **B 342** (1990) 486.

[24] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B 371** (1992) 191.

[25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surveys Diff. Geom.* **1** (1991) 243.

[26] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[27] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [hep-th/9207094].

[28] H. Ooguri and C. Vafa, *Worldsheet derivation of a large-N duality*, *Nucl. Phys.* **B 641** (2002) 3 [hep-th/0205297].

[29] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Topological strings and integrable hierarchies*, hep-th/0312085.

[30] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255]; *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106];
M. Aganagic, A. Klemm, M. Marino and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].

[31] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[32] A. Strominger, *Closed string field theory*, *Nucl. Phys.* **B 294** (1987) 93.

[33] M.B. Green and J. Polchinski, *Summing over world sheet boundaries*, *Phys. Lett.* **B 335** (1994) 377 [hep-th/9406012].

[34] S.L. Shatashvili, *On field theory of open strings, tachyon condensation and closed strings*, hep-th/0105076.

[35] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, *Adv. Theor. Math. Phys.* **6** (2003) 403 [hep-th/0111129].

[36] D. Gaiotto, N. Itzhaki and L. Rastelli, *Closed strings as imaginary D-branes*, *Nucl. Phys.* **B 688** (2004) 70 [hep-th/0304192].

[37] R. Gopakumar, *From free fields to AdS*, *Phys. Rev.* **D 70** (2004) 025009 [hep-th/0308184].

JHEP07(2005)053

[38] N. Berkovits, H. Ooguri and C. Vafa, *On the worldsheet derivation of large-N dualities for the superstring*, *Commun. Math. Phys.* **252** (2004) 259 [hep-th/0310118].

[39] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Notes on topological string theory and 2-d quantum gravity*, PUPT-1217 based on lectures given at *Spring school on strings and quantum gravity*, Trieste, Italy, Apr. 24 - May 2, 1990 and at *Cargèse workshop on random surfaces, quantum gravity and strings*, Cargèse, France, May 28 - Jun 1, 1990.

[40] R. Dijkgraaf, *Intersection theory, integrable hierarchies and topological field theory*, hep-th/9201003.

[41] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory*, *JHEP* **02** (2004) 021 [hep-th/0312170].

[42] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333.

[43] E. Brezin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, *Phys. Lett.* **B 236** (1990) 144;
M.R. Douglas and S.H. Shenker, *Strings in less than one-dimension*, *Nucl. Phys.* **B 335** (1990) 635;
D.J. Gross and A.A. Migdal, *Nonperturbative two-dimensional quantum gravity*, *Phys. Rev. Lett.* **64** (1990) 127.

[44] J.M.F. Labastida, M. Pernici and E. Witten, *Topological gravity in two-dimensions*, *Nucl. Phys.* **B 310** (1988) 611.

[45] D. Montano and J. Sonnenschein, *Topological strings*, *Nucl. Phys.* **B 313** (1989) 258.

[46] E.P. Verlinde and H.L. Verlinde, *A solution of two-dimensional topological quantum gravity*, *Nucl. Phys.* **B 348** (1991) 457.

[47] J.L. Harer, *The cohomology of the moduli space of curves*, in *Theory of moduli*, Lecture Notes in Mathematics, Springer-Verlag, Berlin 1988.

[48] K. Strebel, *Quadratic differentials*, Springer-Verlag, Berlin 1984.

[49] S.B. Giddings, E.J. Martinec and E. Witten, *Modular invariance in string field theory*, *Phys. Lett.* **B 176** (1986) 362.

[50] B. Zwiebach, *A proof that Witten's open string theory gives a single cover of moduli space*, *Commun. Math. Phys.* **142** (1991) 193.

[51] N. Seiberg and S.H. Shenker, *A note on background (in)dependence*, *Phys. Rev.* **D 45** (1992) 4581 [hep-th/9201017].

[52] T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Microscopic and macroscopic loops in nonperturbative two-dimensional gravity*, *Phys. Lett.* **B 238** (1990) 279;
G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2-d quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665;
G.W. Moore and N. Seiberg, *From loops to fields in 2-d quantum gravity*, *Int. J. Mod. Phys.* **A 7** (1992) 2601.

[53] J. Ambjørn, J. Jurkiewicz and Y.M. Makeenko, *Multiloop correlators for two-dimensional quantum gravity*, *Phys. Lett.* **B 251** (1990) 517;
J. Ambjørn, L. Chekhov, C.F. Kristjansen and Y. Makeenko, *Matrix model calculations beyond the spherical limit*, *Nucl. Phys.* **B 404** (1993) 127 [hep-th/9302014] erratum *ibid.* **B 449** (1995) 681.

JHEP07(2005)053

[54] J. Distler, *2-d quantum gravity, topological field theory and the multicritical matrix models*, *Nucl. Phys.* **B 342** (1990) 523.

[55] V.A. Kazakov, A.A. Migdal and I.K. Kostov, *Critical properties of randomly triangulated planar random surfaces*, *Phys. Lett.* **B 157** (1985) 295;
D.V. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Analytical and numerical study of the model of dynamically triangulated random surfaces*, *Nucl. Phys.* **B 275** (1986) 641; *Analytical and numerical study of the model of dynamically triangulated random surfaces*, *Nucl. Phys.* **B 275** (1986) 641;
F. David, *Randomly triangulated surfaces in two-dimensions*, *Phys. Lett.* **B 159** (1985) 303;
I.K. Kostov and M.L. Mehta, *Random surfaces of arbitrary genus: exact results for $D = 0$ and -2 dimensions*, *Phys. Lett.* **B 189** (1987) 118.

[56] I.R. Klebanov and R.B. Wilkinson, *Matrix model in two-dimensions and its effective field theory*, *Phys. Lett.* **B 251** (1990) 379.

[57] I.R. Klebanov and R.B. Wilkinson, *Critical potentials and correlation functions in the minus two-dimensional matrix model*, *Nucl. Phys.* **B 354** (1991) 475.

[58] J.D. Edwards and I.R. Klebanov, *Macroscopic boundaries and the wave function of the universe in the $c = -2$ matrix model*, *Mod. Phys. Lett.* **A 6** (1991) 2901.

[59] J.C. Plefka, *Supersymmetric generalizations of matrix models*, `hep-th/9601041`.

[60] H.G. Kausch, *Curiosities at $c = -2$*, `hep-th/9510149`.

[61] V. Gurarie, *Logarithmic operators in conformal field theory*, *Nucl. Phys.* **B 410** (1993) 535 [`hep-th/9303160`].

[62] M.A.I. Flohr, *On fusion rules in logarithmic conformal field theories*, *Int. J. Mod. Phys.* **A 12** (1997) 1943 [`hep-th/9605151`];
M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, *Phys. Lett.* **B 386** (1996) 131 [`hep-th/9606050`];
H.G. Kausch, *Symplectic fermions*, *Nucl. Phys.* **B 583** (2000) 513 [`hep-th/0003029`].

[63] J. Distler and P. Nelson, *Topological couplings and contact terms in 2-d field theory*, *Commun. Math. Phys.* **138** (1991) 273.

[64] J. Distler and P. Nelson, *Semirigid supergravity*, *Phys. Rev. Lett.* **66** (1991) 1955.

[65] J. Distler and P. Nelson, *The dilaton equation in semirigid string theory*, *Nucl. Phys.* **B 366** (1991) 255.

[66] C.M. Becchi, R. Collina and C. Imbimbo, *On the semirelative condition for closed (topological) strings*, *Phys. Lett.* **B 322** (1994) 79 [`hep-th/9311097`]; *A functional and lagrangian formulation of two dimensional topological gravity*, `hep-th/9406096`;
C.M. Becchi and C. Imbimbo, *Gribov horizon, contact terms and Čech-de Rham cohomology in 2d topological gravity*, *Nucl. Phys.* **B 462** (1996) 571 [`hep-th/9510003`].

[67] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [`hep-th/9411210`].

[68] M. Weis, *Topological aspects of quantum gravity*, `hep-th/9806179`;
J. Ambjørn, M.G. Harris and M. Weis, *On the connection between 2d topological gravity and the reduced hermitian matrix model*, *Nucl. Phys.* **B 504** (1997) 482 [`hep-th/9702188`].

[69] I.K. Kostov, B. Ponsot and D. Serban, *Boundary Liouville theory and 2d quantum gravity*, *Nucl. Phys.* **B 683** (2004) 309 [hep-th/0307189].

[70] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality*, *JHEP* **09** (2001) 038 [hep-th/0006240].

[71] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[72] C.B. Thorn, *String field theory*, *Phys. Rept.* **175** (1989) 1;
W. Taylor and B. Zwiebach, *D-branes, tachyons and string field theory*, hep-th/0311017.

[73] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories, I. Foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

[74] M. Bochicchio, *String field theory in the Siegel gauge*, *Phys. Lett.* **B 188** (1987) 330;
C.B. Thorn, *Perturbation theory for quantized string fields*, *Nucl. Phys.* **B 287** (1987) 61.

[75] E. Witten, *On the Kontsevich model and other models of two-dimensional gravity*, IASSNS-HEP-91-24.

[76] C. Itzykson and J.B. Zuber, *Combinatorics of the modular group, 2. The Kontsevich integrals*, *Int. J. Mod. Phys.* **A 7** (1992) 5661 [hep-th/9201001].

[77] S. Dalley, C.V. Johnson, T.R. Morris and A. Watterstam, *Unitary matrix models and 2-d quantum gravity*, *Mod. Phys. Lett.* **A 7** (1992) 2753 [hep-th/9206060].

[78] C.V. Johnson, *On integrable $c < 1$ open string theory*, *Nucl. Phys.* **B 414** (1994) 239 [hep-th/9301112].

[79] C.V. Johnson, *Non-perturbative string equations for type 0A*, *JHEP* **03** (2004) 041 [hep-th/0311129].

[80] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Loop equations and Virasoro constraints in nonperturbative 2-d quantum gravity*, *Nucl. Phys.* **B 348** (1991) 435.

[81] J.A. Shapiro and C.B. Thorn, *Brst invariant transitions between closed and open strings*, *Phys. Rev.* **D 36** (1987) 432; *Closed string-open string transitions and Witten's string field theory*, *Phys. Lett.* **B 194** (1987) 43;
B. Zwiebach, *Interpolating string field theories*, *Mod. Phys. Lett.* **A 7** (1992) 1079 [hep-th/9202015];
A. Hashimoto and N. Itzhaki, *Observables of string field theory*, *JHEP* **01** (2002) 028 [hep-th/0111092];
D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, *Adv. Theor. Math. Phys.* **6** (2003) 403 [hep-th/0111129].

[82] E. Witten, *Overview of K-theory applied to strings*, *Int. J. Mod. Phys.* **A 16** (2001) 693 [hep-th/0007175].

[83] A. Sen, *Open-closed duality: lessons from matrix model*, *Mod. Phys. Lett.* **A 19** (2004) 841 [hep-th/0308068].

[84] J. Teschner, *On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories*, *JHEP* **04** (2004) 023 [hep-th/0308140].